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LINEAR ALGEBRA

Chapter 1: Vector Space and Its Basic Properties

1.1 Definitions 1
1.2 Properties on Vector Space 2
1.3 Propositions on Subspaces 2
1.4 Propositions on Direct Sum and Quotient Space 5
1.5 Properties of Linearly Independent and Dependent and Basis 7
1.6 Counting of Ordered Linearly Independent Set and Subspace 10
1.7 Examples on Vector Spaces and Their Dimension 11
1.8 Subspaces and Their dimension 18

Chapter 2: Linear Transformation and Its Properties

2.1 Definitions 25
2.2 Kinds of Linear Transformation 27
2.3 Some Important Linear Operators 28
2.4 Properties on Linear Transformations 30
2.5 System of Linear Equations 38

Chapter 3: Matrices and Their Properties

3.1 Definitions 41
3.2 Special Matrices 42
3.3 Properties of Matrices 43
3.4 Properties of Determinants 45
3.5 Properties of Inverse of a Matrix 46
3.6 Rank of a Matrix 47
3.7 Properties of Trace of a Matrix 49

Chapter 4: Eigen Values and Eigen Function

4.1 Definition 51
4.2 Similarity of Matrices and Diagonalization 55
4.3 Important Linear Operator / Matrices 51
4.4 Canonical Forms 55
4.5 Jordan Canonical Form 55

Chapter 5: Inner Product and Its Properties

5.1 Definition 66

5.2 Inner Product Space 6
5.3 Some Important Definitions 6
5.4 Quick Review on Inner Product Spaces 7

Chapter 6: Bilinear Form and Quadratic Form

6.1 Bilinear Form 7
6.2 Quadratic Form 8
6.3 Definite, Semi-Definite and Indefinite Quadratic Form 8
6.4 Criterion For Positive Definiteness of A Real Quadratic Form 8
6.5 Classification 8
Assignment Sheet - 1 8
Assignment Sheet - 2 9
Assignment Sheet - 3 10
Assignment Sheet - 4 10
Assignment Sheet - 5 12
Assignment Sheet - 6 12

.... 6
.... 6
.... 7
.... 7
.... 8
.... 8
.... 8
.... 8
.... 8
.... 9
.... 10
.... 10
.... 12
.... 12

CHAPTER 1

VECTOR SPACE AND ITS BASIC PROPERTIES

1.1. Definitions

Internal Composition: Let A be a set. Then the mapping $f : A \times A \to A$ is said to be internal composition and it also called vector addition.

Example: Consider $\mathbb{R}$, the set of all real numbers

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined as: $f((a,b)) = ab \ \forall (a,b) \in \mathbb{R} \times \mathbb{R}; a, b \in \mathbb{R}$. Then f is an internal composition in $\mathbb{R}$.

External Composition: Let V and F be any two non-empty sets, then the mapping $f : V \times F \to V$ is said to be an external composition in V over F. Also called scalar multiplication

Vector Spaces: Let $(F, +, .)$ be the given field then the elements of F are called scalars. And let V, be a non-empty set, then the elements of V are called vectors.

V is said to be a vector space over the field F then the elements of F are called scalars where as if the following postulates are satisfied under vector addition '+' and scalar multiplication '•'

1. $(V,+)$ is abelian group
2. $\forall a \in F$ and $\forall x \in V, ax \in V$. Closed under scalar multiplication
3. $a(a+y) = ax + ay \forall a \in F$ and $\forall x, y \in V$
4. $(a+b)x = ax + bx \qquad \forall a, b \in F$ and $\forall x \in V$
5. $(ab)x = a(bx), \forall a, b \in F, \& x \in V$
6. $1.x = x), \forall x \in V$

Note:

- 1 is the unity element in the field F
- 0 is the zero element of V and 0 is the zero element of F.
- The vector space of V over the field F is denoted as $V(F)$.
- $V(F)$ is a real vector space if F is the field $\mathbb{R}$ of real numbers.
- $V(F)$ is a rational vector space if F is the field $\mathbb{Q}$ of rational numbers.
- $V(F)$ is a complex vector space if F is the field $\mathbb{C}$ of complex numbers.

1.2. Properties on Vector Space

1. If x, y, and z are vectors in a vector space V such that $x+z=y+z$, than $x=y$
2. In any vector space V, the following statements are true
   - **a.** $0x=0$ for each $x\in V$
   - **b.** $(-a)x=-(ax)=a(-x)$ for each $\alpha\in F$ and each $x\in V$.
   - **c.** $a0=0$ for each $\alpha\in F$.

Subspace: A subset W of a vector space V over field F is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

Improper and Proper Subspaces: If V is any vector space then V itself is a subspace of V. The subset consisting of the zero vector alone i.e. $\{\mathbf{0}\}$ is a subspace of V always, called the zero subspace of V. These two subspaces are called Improper subspaces. The subspaces other than $\{\mathbf{0}\}$ and V are called proper subspaces.

Tests for Showing Subspaces

Two-Step Test: A subset W of a vector space $V(F)$ is a sub space of V iff

1. W is non-empty
2. $\forall x, y\in W \Rightarrow x - y\in W$
3. $\forall a\in F, x\in W \Rightarrow ax\in W$

One-Step Test: A non-empty subset W of a vector space $V(F)$ is a subspace of $V(F)$ iff $\alpha x+\beta y\in W.\ \forall\alpha,\beta\in F$ and $\forall x, y\in W$

1.3. Propositions on Subspaces

1. Let V be a vector space over F. A nonempty subset W of V is a subspace of V if and only if $\alpha u+\beta \upsilon\in W$ for all $\alpha,\beta\in F$, whenever $u, \upsilon\in W$
2. Any intersection of subspaces of a vector space V is a subspace of V
3. Let W_1 and W_2 be subspaces of a vector space V then $W_1\cup W_2$ is a subspace of V if and only if $W_1\subseteq W_2$ or $W_2\subseteq W_1$
4. If S_1 and S_2 are nonempty subsets of a vector space V, then the sum of S_1 and S_2, denoted by S_1+S_2 is define as $\{x+y : x\in S_1 \text{ and } y\in S_2\}$
5. Let W_1 and W_2 be subspaces of a vector space V then the sum W_1+W_2 is a subspace of V that contains both W_1 and W_2
6. Any subspace of V that contains both W_1 and W_2 must also contain W_1+W_2 i.e., W_1+W_2 is the smallest subspace contains both W_1 and W_2

Direct Sum: If W_1 & W_2 be two subspaces of a vector space $V(F)$ then the internal direct sum of W_1 & W_2 is denoted by $W_1\oplus W_2$ and each element of $W_1\oplus W_2$ can be expressed as a sum of unique elements of W_1 & W_2. i.e. If every element of V can be expressed as a sum of unique element of W_1 & W_2 then $V = W_1\oplus W_2$

Note: If W_1 & W_2 be two subspaces of a vector space $V(F)$ such that $V = W_1 \oplus W_2$ then W_1 & W_2 are called complement to each other.

Theorem: A vector space V is called the Direct sum of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. V is the direct sum of W_1 and W_2 then we writing $V = W_1 \oplus W_2$

Example: Consider the vector space $V = \mathbb{R}^2$ over real field $\mathbb{R}$ and $W_1 = \{(x,0) \mid x \in \mathbb{R}\}$, $W_2 = \{(0,y) \mid y \in \mathbb{R}\}$, $W_3 = \{(x,x) \mid x \in \mathbb{R}\}$ are three subspace of V then $V = W_1 \oplus W_2$, $V = W_2 \oplus W_3$ and also $V = W_1 \oplus W_3$

Note: Complement Subspace may or may not be unique .

Orthogonal Complement Subspace: If W_1 & W_2 be two subspaces of a vector space $V(F)$ such that $V = W_1 \oplus W_2$ then W_1 are called Orthogonal complement of W_2 subspace if each vector of W_1 is orthogonal to each vector of W_2 .

Example:

1. Let $V = \mathbb{R}^2$ and let $W = \{(a,\ 0) : a \in R\}$, $W_1 = \{(0,\ b) : b \in R\}$ and $W_2 = \{(c,\ c) : c \in R\}$ then $V = W \oplus W_1$ and $V = W \oplus W_2$.

   Note: Subspace W and W_1 are orthogonal complement subspace. W_1 and W_2 are complement subspace but not orthogonal.

2. Let $V = \mathbb{R}^3(\mathbb{R})$ and let $W_1 = \{(x,\ y,\ 0) : x\ ,\ y \in \mathbb{R}\}$ and $W_2 = \{(x,\ 0,\ z) : x,\ z \in \mathbb{R}\}$ be two subspaces of V. then $V = W_1 + W_2$. But V is not the direct sum of W_1 and W_2 .

3. If V is a vector space of all real valued function over $\mathbb{R}$ then it is a direct sum of its subspace of even and odd functions.

4. A vector space of all $n \times n$ matrices is direct sum of subspace of symmetric and skew symmetric matrices.

5. A vector space of all $n \times n$ matrices with complex entries is direct sum of subspace of Hermitian and skew Hermitian matrices

6. If M, N_1 and N_2 are three subspaces of a vector space V such that

7. $M \oplus N_1 = M \oplus N_2 = V, \not\Rightarrow N_1 \neq N_2$

   Example: Let $V = \mathbb{R}^3$ and $M = \{(a,b,0) : a,b \in \mathbb{R}\}$, $N_1 = \{(0,b,b) : b \in \mathbb{R}\}$, $N_2 = \{(0,0,c) : c \in \mathbb{R}\}$

8. $V = \mathbb{R}[x]$ Polynomial space over $\mathbb{R}$ and $W_1 = \{p(x) \in V \mid p(0) = 0, p'(0) = 0, \ldots, p^{(k)}(0) = 0\}$ where $p^{(k)}(x)$ denotes the k^{th} derivative of $p(x)$. $W_2 = \{p(x) \in V \mid \deg(p(x)) \le k\}$ then $V = \mathbb{R}[x]$ is direct sum of W_1 & W_2 .

9. $V=\mathbb{R}^n$ be a vector space over field $\mathbb{R}$ then $W_1=\left\{(a_1,a_2,\dots,a_n)\,|\,a_i=0, if\ i=1,2,\dots,\left[\frac{n}{2}\right]\right\}$ and

$W_2=\left\{(a_1,a_2,\dots,a_n)\,|\,a_i=0, if\ i=\left[\frac{n+1}{2}\right],\dots,n\right\}$ then V is direct sum of W_1 and W_2 iff n is odd

Cosets of Subspace: Let W be a subspace of a vector space V over a field F. For any $v\in V$ the set $v+W=\{v+w\,|\,w\in W\}$ is called the coset of W containing v. Denote this coset by $v+W$ rather than $v+W$.

Quotient Space: Let W be a subspace of a vector space V over a field F and $S=\{v+W : v\in V\}$ of all cosets of W then set S is a vector space over a field F with the operations internal composition

$(v_1+W)+(v_2+W)=(v_1+v_2)+W$ and external composition is $a(v+W)=av+W$. This vector space is called the quotient space of V modulo W and is denoted by $\frac{V}{W}$

Propositions:

1. Let W be a subspace of a finite dimensional vector space $V(F)$. Then

$$dim\left(\frac{V}{W}\right)=\dim(V)-\dim(W)$$

2. Let W be a finite dimensional subspace of a infinite dimensional vector space $V(F)$. Then $dim\left(\frac{V}{W}\right)$ = infinite dimension

3. Let W be aninfinite dimensional subspace of aninfinite dimensional vector space $V(F)$. Then $dim\left(\frac{V}{W}\right)$ can be infinite dimension or finite dimension

Example:

1. Let V be the vector space of all 2 × 2 matrices over $\mathbb{C}$. And $w_1=\left\{\begin{bmatrix} x & y \\ z & 0\end{bmatrix} : x,y,z\in\mathbb{C}\right\}$, $w_2=\left\{\begin{bmatrix} x & 0 \\ 0 & y\end{bmatrix} : x,y\in\mathbb{C}\right\}$, then

$$\dim\left(\frac{w_1+w_2}{w_2}\right)=\dim\left(\frac{w_1}{w_1\cap w_2}\right)=2$$

2. Let V the vector space of all 2 × 2 matrices over $\mathbb{C}$. $w=\left\{\begin{bmatrix} x & -x \\ y & z\end{bmatrix} : x,y,z\in\mathbb{C}\right\}$, $w_2=\left\{\begin{bmatrix} a & b \\ -a & c\end{bmatrix} : a,b,c\in\mathbb{C}\right\}$ then

$$\dim\left(\frac{w_1+w_2}{w_2}\right)=\dim\left(\frac{w_1}{w_1\cap w_2}\right)=1$$

4. Propositions Direct Sum and Quotient Space

1. Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$
2. $v + W$ is a subspace of V if and only if $v \in W$.
3. $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.
4. Let W be a subspace of a vector space V over a field F, $S = \{v + W : v \in V\}$ of all cosets of W then $a(v + W) = av + W$

Spanning of Vector Spaces: We're interested is spanning down what it means for a vector space to have a basis, and that's described in terms of the concept of linear combination. Span and independence are two more related concepts.

Linear Combination: a linear combination is an expression constructed from a set of terms by multiplying each term by a constant and adding the results (e.g. a linear combination of x and y would be any expression of the form $ax + by$, where a and b are constants). The concept of linear combinations is central to linear algebra and related fields of mathematics.

Let $V(F)$ be a vector space over field F and $S \subseteq V$ be a subset of V may be finite or infinite then for any vectors $x_1, x_2, \ldots\ldots\ldots, x_n \in S$ and scalars $\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n \in F$ implies $x = \sum_{i=1}^{n} \alpha_i x_i \in V$ is defined as linear combination of vectors $x_1, x_2, \ldots\ldots\ldots, x_n \in S$ or x is in linear combination of vectors $x_1, x_2 \ldots, x_n$

Example: Let $S \subset V(F)$ where $S = \{x_1, x_2\}$ and F is finite field say $F = \{\alpha_1, \alpha_2, \alpha_3\}$, then $\alpha_1 x_1$, $\alpha_2 x_2, \alpha_3 x_1, \alpha_1 x_1 + \alpha_2 x_2$ etc. are different linear combination of the elements of S.

Linear Span or Spanning Set or Generating Set: The linear span of a non-empty subset S of Vector space $V(F)$ is defined as the set consisting of all linear combinations of the elements of S and denoted by $L(S)$. or $< S >$ or $Span(S)$

OR

"set of all possible linear combinations of the elements of S is called the linear span of S"

In fact $L(S)$ is the smallest subspace of V containing S in other words we can say that $L(S)$ is the intersection of all the subspaces containing S.

Theorem: Let $V(F)$ be a vector space and $S \subseteq V$ then $L(S)$ is subspace of V called subspace spanned by S or generated by S.

Note: As ϕ is the subset of all the vector subspace of V and if we take intersection of all the subspaces of V the resulting subspace is $\{\mathbf{0}\}$ hence we can say $L(\phi) = \{0\}$

Example: Let V be the vector space of all polynomial function over F. Let S be the subset of V consisting of the polynomial functions $f_0, f_1, f_2, \ldots\ldots\ldots$ defined by $f_n(x) = x^n,\ n = 0,1,2\ldots\ldots$ then V is the subspace spanned by the set S.

Linear Dependence and Independence

Linear Dependence: Let V be a vector space over F. A subset S of V is said to be dependent if there exists distinct vectors $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ in S and scalar $c_1, c_2, \ldots c_n$ in F, not all of which are 0, such that $c_1x_1 + c_2x_2 + -------- + c_nx_n = 0$

If the set S contains finitely many vectors $x_1, x_2, \ldots\ldots. x_n$ then sometimes we can say that the vectors $x_1, x_2, \ldots\ldots\ldots., x_n$ are dependent.

In other words if $x_1, x_2, \ldots\ldots\ldots, x_n$ are linearly dependent. Then $\exists c_1, c_2 \ldots\ldots\ldots, c_n$ not all zero s.t. $c_1x_1 + c_2x_2 --- -------- + c_nx_n = 0$ Without loss of generality we can assume $c_1 \neq 0$ then $x_1 = \left(-c_1^{-1}c_2\right) x_2 + \left(-c_1^{-1}c_3\right) x_3 + \ldots\ldots + \left(-c_1^{-1}c_n\right) x_n$, hence we can say x_1 is linear combinations of the others.

Linear Independent: Any set containing the vectors $x_1, x_2, \ldots\ldots, x_n$ defined over a field F is said to be linearly independent if it is not linear dependent i.e. if $c_1x_1 + c_2x_2 + \ldots\ldots\ldots + c_nx_n = 0$

Implies $c_i = 0$ for all i.

Example: In $\mathbb{R}^3$ the vectors $x_1 = (3,0,-3)$, $x_2 = (-1,1,2)$, $x_3 = (4,2,-2), x_4 = (2,1,1)$ are linearly dependent since $2\,x_1 + 2\,x_2 - x_3 + 0x_4 = 0$

Example: Let V be the vector space of 2×2 matrices over $\mathbb{R}$. Determine whether the matrices $A, B, C \in V$ are dependent where:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} 1$$

Example: Let V be the vector space of functions from $\mathbb{R}$ into $\mathbb{R}$. Show that $f, g, h \in V$ are independent where: $f(t) = e^{2t}, g(t) = t^2, h(t) = t$; (ii) $f(t) = \sin t, g(t) = \cos t, h(t) = t$.

Vector Space Spanned by Subset: $V(F)$, be a Vector Space and $S \subseteq V$ then V is called spanned by S if $L(S) = V$

Finitely Generated Vector Space (FGVS) and Non-Finitely Generated Vector Space (NFGVS): A Vector Space $V(F)$, is called finitely generated vector space if there exists finite $S \subseteq V$ such that $L(S) = V$ other wise non-finitely Generated Vector Space (NFGVS)

Example: $V = \mathbb{R}^3$ is vector space over $\mathbb{R}$ is finitely generated vector space. As $S = \{(1,0,0),(0,1,0),(0,0,1)\} \subseteq \mathbb{R}^3$ spanned $V = \mathbb{R}^3$

1. $\mathbb{R}(\mathbb{Q})$ is non-finitely generated vector space (NFGVS)
2. $V = \mathbb{R}[x]$ is polynomial space over field $\mathbb{R}$ is non-finitely generated vector space (NFGVS)
3. $V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is function }\}$ then V is vector space over field $\mathbb{R}$. Under addition of functions and scalar multiplication to the functions. $V(\mathbb{R})$ is non-finitely generated vector space (NFGVS)

Basis and Dimension: Let V be a Vector Space, a basis for V is a linearly independent set of vectors in V which spans the space V. The space V is finite dimensional if it has a finite basis i.e. the basis has finite number of elements. i.e., let $V \subset V(F)$, then S is a basis if

1. S is linearly independent
2. $L(S) = V(F)$

Dimension of Vector Space: Dimension of a finite dimensional vector space is the number of vectors in any basis set of the vector space and it is denoted by $dim(V)$

Example:

1. $\mathbb{C}(\mathbb{C})$ is a vector space of dimension one as $S = \{1\}$ is Linearly independent set which spans $\mathbb{C}(\mathbb{C})$
2. $\mathbb{C}(\mathbb{R})$ is a vector space of dimension two as $S = \{1, i\}$ is linearly independent set which spans $\mathbb{C}(\mathbb{R})$
3. $V = \mathbb{R}[x]$ is polynomial space over field $\mathbb{R}$ is infinite dimension vector space .
4. $V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is function }\}$ then V is vector space over field $\mathbb{R}$. Under addition of functions and scalar multiplication to the functions. $V(\mathbb{R})$ is infinite dimension vector space .

Ordered Bases: If V is a finite dimensional vector space, then an ***ordered basis*** for V is a finite sequence of vectors which is linearly independent and spans V.

i.e. let $S = \{x_1, x_2, \ldots\ldots\ldots, x_n\}$ be a basis of $V(F)$ then it is called an ordered basis if the position of vectors in S is fixed i.e. the set S is ordered, hence if $S = \{x_1, x_2, x_3\}$ is an ordered basis then the basis $\{x_2, x_1, x_3\}$ is considered as different ordered basis.

5. Properties on Linearly Independent and Dependent and Basis

1. Every subset of linearly independent set is linearly independent. That is Let $V(F)$ be a vector space, and let $S_1 \subseteq S_2 \subseteq \mathrm{V}$. If S_2 is linearly independent, then S_1 is linearly independent
2. The empty set is linearly independent; hence every linearly dependent sets must be nonempty.
3. A set consisting of a single nonzero vector is linearly independent. For if $\{u\}$ is linearly dependent, then $au = 0$ for some nonzero scalar a
4. A set linearly independent if and only if the only representations of 0 as linear combinations of its vectors are trivial representations

5. A set linearly dependent if and only if the only representations of 0 as linear combinations of its vectors are non-trivial representations
6. Any set containing 0 is linearly dependent.
7. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent
8. Let S be linearly independent subset of a vector space and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \operatorname{span}(S)$
9. Let u and v be district vectors in vector space V. then $\{u,v\}$ is linearly dependent if and only if u or v is a multiple of the other
10. Let V be a vector space over a field of characteristic not equal to two.
    - **a.** Let u and u be distinct vectors in V. then $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent.
    - **b.** Let u, v and w be distinct vectors in V. Then $\{u,v,w\}$ is linearly independent if and only if $\{u+v, u+w, v+w\}$ in linearly independent
11. Let $S=\{u_1,u_2,...,u_n\}$ be a finite set of vectors. Then S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \operatorname{span}(\{u_1,u_2,...,u_k\})$ for some $k(1 \le k < n)$
12. A set S of vectors is linearly independent if and only if each finite subset of S is linearly independent
13. S be a set of nonzero polynomials in $P(F)$ such that the same degree. then S is linearly independent
14. if $\{A_1, A_2,..., A_k\}$ is a linearly independent subset of $M_{n\times n}(F)$, then $\{A_1^t, A_2^t,..., A_k^t\}$ is also linearly independent
15. Let $f, g, \in F(\mathbb{R},\mathbb{R})$ be the functions defined by $f(t)=e^{rt}$ and $g(t)=e^{st}$, where $r \ne s$. Then that f and g are linearly independent in $F(\mathbb{R},\mathbb{R})$.
16. Let V be a vector space and $\beta=\{u_1,u_2,...,u_n\}$ be a subset of V. Then β is a basis for V. If and only if each $v \in V$ can be uniquely expressed as linear combination of vectors of β, that is, can be expressed in the form $v=a_1u_1+a_2u_2+\cdots+a_nu_n$
17. If a vector space V is generated by a finite set S, then some subset of S is a basis for V, i.e. V has a finite basis
18. A vector space V over a field K is said to be finite dimensional if there are finite number of elements $x_1,...,x_n$ in V such that $V=\langle x_1,...x_n\rangle$. If no such finite number of elements exist in V, V is called infinite dimensional

19. If vector space of continuously differentiable functions is taken then any subset of this vector space is linearly independent if ***Wronskian*** of the functions is none zero.

20. Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors

21. Let V be a vector space with dimension n

- **a.** Any finite generating set for V contains at least n vectors, and generating set V that contains exactly n vectors is a basis for V .
- **b.** Any linearly independent subset of V that contains exactly n vectors is a basis for V .
- **c.** Every linearly independent subset of V can be extended to a basis for V .

22. Let W be a subspace of finite-dimensional vector space V . Then W is finite-dimensional and dim (W) $\dim(W) \le \dim(V)$. Moreover, if $\dim(W)=\dim(V)$, then $V=W$

23. If W is a subspace of finite-dimensional vector space V , then any basis for W can be extended to basis for V

24. if W_1 and W_2 are finite-dimensional subspaces of a vector space V , then

- **a.** The subspace W_1+W_2 is finite-dimensional, and
- **b.** W_1+W_2 is smallest subspace of V containing $W_1 \cup W_2$ i.e. $L(W_1 \cup W_2)=W_1+W_2$
- **c.** $\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)-\dim(W_1 \cap W_2)$

25. Let W_1 and W_2 be subspaces of vector space V having dimensions m and n, respectively, where $m \ge n$ Then

- **a.** $\dim(W_1 \cap W_2) \le n$
- **b.** $\dim(W_1+W_2) \le m+n$

26. Let W_1 and W_2 be subspaces of a vector space V such that $V=W_1 \oplus W_2$

- **a.** If β_1 and β_2 are bases for W_1 and W_2 , respectively, then $\beta_1 \cap \beta_2 = \phi$ and $\beta_1 \cup \beta_2$ is a basis for V
- **b.** let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2 , respectively, of a vector space V if $\beta_1 \cup \beta_2$ is a basis for V , then $V=W_1 \oplus W_2$

27. Let W be a subspace of a (not necessarily finite-dimensional) vector space V then any basis for W is a subset of basis for V

28. let V be a finite dimensional vector space over field K , and let X and Y be finite subsets of V If Y is linearly independent and $V=\langle X \rangle$, then $|Y| \le |X|$

29. Let V be finite dimensional vector space over field K . If X is a linearly independent subset of V is finite

30. Let V be finite dimensional vector space over K. Then the following statements are equivalent for a subset β of V :

a. β is a basis

b. β is a minimal generating set, that is, no subset of β can generate V

c. β is a maximal linearly independent set

31. Let V be a vector space of dimension n. Then every linearly independent subset of V with n elements is a basis of V

32. $\frac{V}{W} \subset P(V)$ but never contained in V.

33. If x_i's are linearly independent in V then it is not necessary that $(W + x_i)$'s are linearly independent

34. If x_i's are linearly dependent in V then $(W + x_i)$'s are also linearly dependent in $\frac{V}{W}$.

35. For every FDVS V there exists subspaces W_1 and W_2 s.t. $V = W_1 \oplus W_2$

36. $\dim(W_1 \cap W_2) \le \min\left[\dim(W_1), \dim(W_2)\right]$

37. $\dim(W_1 \cap W_2) \le \max\left[0, \dim(W_1) + \dim(W_2) - \dim(V)\right]$

38. $\max[\dim(W_1), \dim(W_2)] \le \dim(W_1 + W_2) \le \min[\dim(W_1) + \dim(W_2), \dim(V)]$

39. If $P_n(x, y)$ denotes vector space of polynomial with degree less than or equal to n in indeterminate x and y then following is true

a. $\dim(P_n(x, y)) = \frac{(n+1)(n+2)}{2!}$

b. $\dim(P_n(x, y, z)) = \frac{(n+1)(n+2)(n+3)}{3!}$ and can be generalized.

1.6. Counting of Ordered Linearly Independent Set and Subspace

If $\dim V(F) = n$ and $Card(F) = p^k$ where n is dimension of V over F and F is finite. then

1. $|V(F)| = p^{nk}$ i.e. $|V(F)| = |F|^n$

2. $Number\ of\ unordered\ basis = \frac{Number\ of\ ordered\ basis}{n!}$

3. The number of linearly independent subsets of cardinality l are $= \frac{\left(p^{nk} - 1\right)\left(p^{nk} - p\right)......\left(p^{nk} - p^{nl-1}\right)}{k!}$

4. The number of ordered linearly independent subsets of cardinality k are. The number of basis of $V = \left(p^n - 1\right)\left(p^n - p\right)............\left(p^n - p^{k-1}\right)$

5. The number of basis of $V = \dfrac{(p^n-1)(p^n-p)\ldots\ldots\ldots\ldots(p^n-p^{n-1})}{n!}$

6. The number of ordered basis of $V = (p^n-1)(p^n-p)\ldots$ $\ldots\ldots\ldots(p^n-p^{n-1})$

Note: Number of ordered basis is actually the order of the group $GL(n,F)$

1. The number of subspaces of dimension k of V $= \dfrac{(p^n-1)(p^n-p)\ldots\ldots\ldots\ldots(p^n-p^{k-1})}{(p^k-1)(p^k-p)\ldots\ldots\ldots\ldots(p^k-p^{k-1})}$

2. Number of subspaces of $V(F)$,defined as above ,with dimension r $= \dfrac{\prod_{i=0}^{r-1}(p^{nk}-p^{ki})}{\prod_{i=0}^{r-1}(p^{kr}-p^{ki})}$

3. Let $|K| = p^r$ and $|F| = p^n$ and $K < F$ then r/n.

If V is n-dimensional vector space over field F, where $|F| = p$, then

1.7. Examples on Vector Spaces and Their Dimension

1. Let $V(F)$ denotes the V is a Vector space over field F then which of the following is a Vector space under usual additions of vectors and scalar multiplication?

2. Every field is Vector space over its subfield is a vector space, i.e. if F' is subfield of F then $F(F')$ is a Vector space

Structure	Vector Space	Dimension
1. $\mathbb{C}(\mathbb{C})$	Yes	1
2. $\mathbb{C}(\mathbb{R})$	Yes	2
3. $\mathbb{C}(\mathbb{Q})$	Yes	Infinite dimension
4. $\mathbb{C}(\mathbb{Q}(\sqrt{2}))$	Yes	Infinite dimension
5. $\mathbb{R}(\mathbb{C})$	NO	-
6. $\mathbb{R}(\mathbb{R})$	Yes	1
7. $\mathbb{R}(\mathbb{Q})$	Yes	Infinite dimension

8. $\mathbb{R}(\mathbb{Q}(\sqrt{2}))$	Yes	Infinite dimension
9. $\mathbb{R}(\mathbb{Q}^c)$	No	-
10. $\mathbb{Q}(\mathbb{C})$	No	-
11. $\mathbb{Q}(\mathbb{R})$	No	-
12. $\mathbb{Q}(\mathbb{Q})$	Yes	1
13. $\mathbb{Q}(\mathbb{Q}(\sqrt{2}))$	No	-
14. $\mathbb{Q}(\mathbb{Q}^c)$	No	-
15. $\mathbb{Q}(\sqrt{2})(\mathbb{C})$	No	-
16. $\mathbb{Q}(\sqrt{2})(\mathbb{R})$	No	-
17. $\mathbb{Q}(\sqrt{2})(\mathbb{Q})$	Yes	2
18. $\mathbb{Q}(\sqrt{2})(\mathbb{Q}(\sqrt{2}))$	Yes	1

3. Let F' is subfield of F and let $F^n = \{(a_1, a_2, \ldots a_n,) \mid a_i \in F\}$ set of n-tuple's of field then $F^n(F')$ is Vector Space

$\mathbb{C}^n$		
Over	**Vector Space**	**Dimension**
1. $\mathbb{C}$	Yes	n
2. $\mathbb{R}$	Yes	2n
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension
$\mathbb{R}^n$		
Over	**Vector space**	**Dimension**

1.	$\mathbb{C}$	No	-
2.	$\mathbb{R}$	Yes	n
3.	$\mathbb{Q}$	Yes	Infinite dimension
4.	$\mathbb{Q}^c$	No	-
5.	$\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension

$\mathbb{Q}^n$			
Over		**Vector space**	**Dimension**
1.	$\mathbb{C}$	No	-
2.	$\mathbb{R}$	No	-
3.	$\mathbb{Q}$	Yes	n
4.	$\mathbb{Q}^c$	No	-
5.	$\mathbb{Q}(\sqrt{2})$	No	-

$\mathbb{Q}(\sqrt{2})^n$			
Over		**Vector Space**	**Dimension**
1.	$\mathbb{C}$	No	-
2.	$\mathbb{R}$	No	-
3.	$\mathbb{Q}$	Yes	2n
4.	$\mathbb{Q}^c$	No	-
5.	$\mathbb{Q}(\sqrt{2})$	Yes	1n

4. Let F' is subfield of F and let $V = F^{n\times m} = \left\{A = \left[a_{ij}\right] \mid a_{ij} \in F\right\}$ set of $n \times m$ matrices over field F, $V(F')$ is vector space over field F' under matrix addition and scalar multiplication

$\mathbb{C}^{n\times m}$		
Over	**Vector space**	**Dimension**

1. $\mathbb{C}$	Yes	$m.n$
2. $\mathbb{R}$	Yes	$2m.n$
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension
$\mathbb{R}^{n\times m}$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	Yes	$m.n$
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension
$\mathbb{Q}^{n\times m}$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	No	-
3. $\mathbb{Q}$	Yes	$m.n$
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	No	-
$\mathbb{Q}(\sqrt{2})^{n\times m}$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	No	-

3. $\mathbb{Q}$	Yes	$2m.n$
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	$m.n$

Put Your Own Notes

5. Let F' is subfield of F and let $F[x]=\{P(x)|P(x)$ is a polynomial with coefficients from field $F\}$,then $F[x]$ is Vector Space over Field F' under addition of polynomials and Scalar multiplication

$\mathbb{C}[x]$

Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	Yes	Infinite dimension
2. $\mathbb{R}$	Yes	Infinite dimension
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension

$\mathbb{R}[x]$

Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	Yes	Infinite dimension
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension

$\mathbb{Q}[x]$

Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	No	-
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-

1. $\mathbb{C}$	Yes	$m.n$
2. $\mathbb{R}$	Yes	$2m.n$
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension
$\mathbb{R}^{n\times m}$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	Yes	$m.n$
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension
$\mathbb{Q}^{n\times m}$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	No	-
3. $\mathbb{Q}$	Yes	$m.n$
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	No	-
$\mathbb{Q}(\sqrt{2})^{n\times m}$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	No	-

3. $\mathbb{Q}$	Yes	$2m.n$
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	$m.n$

5. Let F' is subfield of F and let $F[x]=\{P(x)\,|\,P(x)$ is a polynomial with coefficients from field $F\}$,then $F[x]$ is Vector Space over Field F' under addition of polynomials and Scalar multiplication

$\mathbb{C}[x]$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	Yes	Infinite dimension
2. $\mathbb{R}$	Yes	Infinite dimension
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension
$\mathbb{R}[x]$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	Yes	Infinite dimension
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension
$\mathbb{Q}[x]$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	No	-
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-

5. $\mathbb{Q}(\sqrt{2})$	No	-

6. Let F' is subfield of F and $I=[0,1]$ be an interval containing 0, let $F(I)=\{f \mid f$ is a function from I to $F\}$, then $F(I)$ is Vector Space over Field F' under addition of functions and Scalar multiplication.

7. Which of the following is a Vector space under addition of functions and scalar multiplication of functions

$\mathbb{C}(I)$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	Yes	Infinite dimension
2. $\mathbb{R}$	Yes	Infinite dimension
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension
$\mathbb{R}(I)$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	Yes	Infinite dimension
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension
$\mathbb{Q}(I)$		
Over	**Vector space**	**Dimension**
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	No	-
3. $\mathbb{Q}$	Yes	Infinite dimension

4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	No	-

$\mathbb{Q}(\sqrt{2})(I)$

Over	Vector space	Dimension
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	No	-
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension

8. Let F' is subfield of F let $F^{<a_n>} = \{<a_n> \mid <a_n> \text{ is a sequence on } F\}$,then $F^{<a_n>}$ is Vector Space over Field F' under addition of sequence and Scalar multiplication

9. Which of the following is a Vector space under addition of sequence and scalar multiplication of sequence

$\mathbb{C}^{<a_n>}$

Over	Vector Space	Dimension
1. $\mathbb{C}$	Yes	Infinite dimension
2. $\mathbb{R}$	Yes	Infinite dimension
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension

$\mathbb{R}^{<a_n>}$

Over	Vector space	Dimension
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	Yes	Infinite dimension

3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension

$\mathbb{Q}^{<a_n>}$

Over	Vector space	Dimension
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	No	-
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	No	-

$\mathbb{Q}(\sqrt{2})^{<a_n>}$

Over	Vector space	Dimension
1. $\mathbb{C}$	No	-
2. $\mathbb{R}$	No	-
3. $\mathbb{Q}$	Yes	Infinite dimension
4. $\mathbb{Q}^c$	No	-
5. $\mathbb{Q}(\sqrt{2})$	Yes	Infinite dimension

1.8. Subspaces Spaces and Their Dimension

1. Which of the following is a subspace of given Vector space

Vector space	$\mathbb{C}(\mathbb{C})$	
Subset	**Subspace**	**Dimension**
1. $\mathbb{C}(\mathbb{C})$	Yes	1
2. $\mathbb{R}(\mathbb{C})$	No	-
3. $\mathbb{Q}(\mathbb{C})$	No	-

4. $\mathbb{Q}(\sqrt{2})(\mathbb{C})$	No	-
5. $\mathbb{C}(\mathbb{R})$	No	-
Vector space	$\mathbb{C}(\mathbb{R})$	
Subset	**Subspace**	**Dimension of subspace**
1. $\mathbb{Q}(\mathbb{R})$	No	-
2. $\mathbb{R}(\mathbb{Q})$	No	-
3. $\mathbb{Q}(\sqrt{2})(\mathbb{R})$	No	-
4. $\mathbb{Q}(\sqrt{2})(\mathbb{Q})$	No	-
5. $\mathbb{R}(\mathbb{R})$	Yes	1
Vector space	$\mathbb{R}(\mathbb{R})$	
Subset	**Subspace**	**Dimension of subspace**
1. $\mathbb{C}(\mathbb{R})$	No	-
2. $\mathbb{R}(\mathbb{R})$	Yes	-
3. $\mathbb{Q}(\sqrt{2})(\mathbb{R})$	No	-
4. $\mathbb{Q}(\mathbb{R})$	No	-

2. Which of the following is a subspace of given Vector space Let F' is subfield of F and let $F^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in F\}$ set of n-tuple's of field then

Vector space	$\mathbb{C}^n(\mathbb{C})$	
Subset	**Subspace**	**Dimension**
$W_1 = \left\{(a_1, a_2 \dots\dots a_n) \mid \sum_{i=1}^{n} a_i = 0\right\}$	**Yes**	$n-1$
$W_2 = \left\{(a_1, a_2 \dots\dots a_n) \mid \sum_{i=1}^{n} \alpha_i a_i = 0, \alpha_i \in F'\right\}$	**Yes**	$n-1$
$W_3 = \left\{(a_1, a_2 \dots\dots, a_n) \mid \sum_{i=1}^{n} i a_i = 0\right\}$	**Yes**	$n-1$

$W_4 = \{(a_1, a_2 \ldots\ldots, a_n) \mid a_{2i} = 0\}$ $i = 1, 2 \ldots \left\lceil \frac{n+1}{2} \right\rceil$	Yes	$\left\lceil \frac{n+1}{2} \right\rceil$
$W_5 = \{(a_1, a_2 \ldots\ldots, a_n) \mid a_{2i-1} = 0\}$ $i = 1, 2 \ldots \left\lceil \frac{n+1}{2} \right\rceil$	Yes	$\left\lceil \frac{n}{2} \right\rceil$
$W_6 = \{(a_1, a_2 \ldots\ldots, a_n) \mid a_i = 0 \ if \ i/n\}$ $i = 1, 2 \ldots n$	Yes	$n - \tau(n)$
$W_7 = \{(a_1, a_2, \ldots, a_n) : \mid a_k \mid \geq 0, 1 \leq k \leq n\}$	No	-
$W_8 = \{(a_1, a_2 \ldots, a_n) \mid a_1 = \lambda a_n,\ \lambda$ fixed $\lambda \in F'\}$	Yes	$n-1$
$W_9 = \left\{(a_1, a_2 \ldots\ldots, a_n) \mid \frac{a_1}{a_n} = \lambda\right.$ fixed, $\left.\lambda \in F'\right\}$	No	-
$W_{10} = \left\{(a_1, a_2 \ldots\ldots, a_n) \mid \sum_{i=1}^{n} (-1)^{i+1} a_i = 0\right\}$	Yes	$n-1$
Vector Space	$\mathbb{C}^n(\mathbb{R})$	
Subset	**Subspace**	**Dimension**
$W_1 = \left\{(a_1, a_2 \ldots\ldots a_n) \mid \sum_{i=1}^{n} a_i = 0\right\}$	Yes	$2(n-1)$
$W_2 = \left\{(a_1, a_2 \ldots\ldots a_n) \mid \sum_{i=1}^{n} \alpha_i a_i = 0, \alpha_i \in F'\right\}$	Yes	$2(n-1)$
$W_3 = \left\{(a_1, a_2 \ldots\ldots, a_n) \mid \sum_{i=1}^{n} i a_i = 0\right\}$	Yes	$2(n-1)$
$W_4 = \{(a_1, a_2 \ldots\ldots, a_n) \mid a_{2i} = 0\}$ $i = 1, 2 \ldots \left\lceil \frac{n+1}{2} \right\rceil$	Yes	$2\left\lceil \frac{n+1}{2} \right\rceil$
$W_5 = \{(a_1, a_2 \ldots\ldots, a_n) \mid a_{2i-1} = 0\}$ $i = 1, 2 \ldots \left\lceil \frac{n+1}{2} \right\rceil$	Yes	$2\left\lceil \frac{n}{2} \right\rceil$

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$W_6 = \{(a_1, a_2 \ldots\ldots, a_n) \mid a_i = 0 \text{ if } i/n\}$ $i = 1, 2 \ldots n$	Yes	$2(n - \tau(n))$
$W_7 = \{(a_1, a_2, \ldots, a_n) : \lvert a_k \rvert \geq 0, 1 \leq k \leq n\}$	No	-
$W_8 = \{(a_1, a_2 \ldots, a_n) \mid a_1 = \lambda a_n \text{ , } \lambda \text{ fixed, } \lambda \in$	Yes	$2(n-1)$
$W_9 = \left\{(a_1, a_2 \ldots\ldots, a_n) \mid \frac{a_1}{a_n} = \lambda \text{ fixed, } \lambda \in F\right.$	No	-
$W_{10} = \left\{(a_1, a_2 \ldots\ldots, a_n) \mid \sum_{i=1}^{n} (-1)^{i+1} a_i = 0\right\}$	Yes	$2(n-1)$

3. Which of the following is a subspace of given Vector space of $F^{n\times m} = \left\{A = \left[a_{ij}\right] \mid a_{ij} \in F\right\}$ set of $n \times m$ matrices over field F. $F^{n\times m}(F')$ is Vector Space over Field F' under matrix addition and Scalar multiplication, where F' is subfield of F

4. Which of the following is a subspace of given Vector space $F^{n\times m}(F')$

Vector Space	$\mathbb{C}^{n\times m}(\mathbb{C})$	
Subset	**Subspace**	**Dimension**
$W_1 = \left\{\left[a_{ij}\right] \mid \sum_{j=1}^{m} a_{ij} = 0, i = 1, 2 \ldots n\right\}$	Yes	$mn - n$
$W_2 = \left\{\left[a_{ij}\right] \mid \sum_{i=1}^{m} a_{ij} = 0, i = 1, 2 \ldots m\right\}$	Yes	$mn - m$
$W_3 = W_1 \cap W_2$	Yes	$(n-1)(m-1)$
$W_4 = W_1 \cup W_2$	No	-
$W_5 = \{\left[a_{ij}\right] \mid a_{ij} = 0 \text{ if } i/n\}$	Yes	$mn - m.\tau(n)$
Vector space	$\mathbb{C}^{n\times m}(\mathbb{R})$	
Subset	**Subspace**	**Dimension**
$W_1 = \left\{\left[a_{ij}\right] \mid \sum_{j=1}^{m} a_{ij} = 0, i = 1, 2 \ldots n\right\}$	Yes	$2(mn - n)$
$W_2 = \left\{\left[a_{ij}\right] \mid \sum_{i=1}^{m} a_{ij} = 0, i = 1, 2 \ldots m\right\}$	Yes	$2(mn - m)$
$W_3 = W_1 \cap W_2$	Yes	$2((n-1)(m-1))$
$W_4 = W_1 \cup W_2$	No	-
$W_5 = \{\left[a_{ij}\right] \mid a_{ij} = 0 \text{ if } i/n\}$	Yes	$2(mn - m.\tau(n))$

Put Your Own Notes

5. Which of the following is a subspace of given Vector space of $F^{n\times n}=\left\{A=\left[a_{ij}\right]\mid a_{ij}\in F\right\}$ set of $n\times n$ matrices over field F, $F^{n}\left(F'\right)$ is Vector Space over Field F' under matrix addition and Scalar multiplication, where F' is subfield of F

Vector Space	$\mathbb{C}^{n\times n}(\mathbb{C})$	
Subset	**Subspace**	**Dimension**
$W_6=\left\{A=\left[a_{ij}\right]_{m\times n}/Trace(A)=0\right\}$	Yes	n^2-1
$W_7=\left\{A=\left[a_{ij}\right]_{m\times n}/A^T=A\right\}$	Yes	$\frac{n(n+1)}{2}$
$W_8=\left\{A=\left[a_{ij}\right]_{m\times n}/A^T=-A\right\}$	Yes	$\frac{n(n-1)}{2}$
$W_9=W_6\cap W_7$	Yes	0
$W_{10}=W_6\cup W_8$	No	....
$W_{11}=\left\{A=\left[a_{ij}\right]_{n\times n}/A \text{ is upper triangle}\right\}$	Yes	$\frac{n(n+1)}{2}$
$W_{13}=\left\{A=\left[a_{ij}\right]_{n\times n}/A \text{ is diagonal}\right\}$	Yes	n
$W_{14}=\left\{A=\left[a_{ij}\right]_{n\times n}/A \text{ is Scalar}\right\}$	Yes	1
$W_{15}=\left\{A=\left[a_{ij}\right]_{n\times n}/A \text{ is tri diagonal}\right\}$	Yes	$3n-1$
Vector Space	$\mathbb{C}^{n\times n}(\mathbb{R})$	
Subset	**Subspace**	**Dimension**
$W_6=\left\{A=\left[a_{ij}\right]_{m\times n}/Trace(A)=0\right\}$	Yes	$2(n^2-1)$
$W_7=\left\{A=\left[a_{ij}\right]_{m\times n}/A^T=A\right\}$	Yes	$n(n+1)$
$W_8=\left\{A=\left[a_{ij}\right]_{m\times n}/A^T=-A\right\}$	Yes	$n(n-1)$
$W_9=W_6\cap W_7$	Yes	0
$W_{10}=W_6\cup W_8$	No	-
$W_{11}=\left\{A=\left[a_{ij}\right]_{n\times n}/A \text{ is upper triangle}\right\}$	Yes	$n(n+1)$

$W_{13}=\left\{A=\left[a_{ij}\right]_{n\times n}/A \text{ is diagonal}\right\}$	**Yes**	$2n$
$W_{14}=\left\{A=\left[a_{ij}\right]_{n\times n}/A \text{ is Scalar}\right\}$	**Yes**	**2**
$W_{15}=\left\{A=\left[a_{ij}\right]_{n\times n}/A \text{ is tri diagonal}\right\}$	**Yes**	$6n-2$

6. Which of the following is a subspace of given Vector space of $F[x]=\{P(x)\,|\,P(x) \text{ is a polynomial with coefficient from field } F\}$, then $F[x]$ is Vector Space over Field F' under addition of polynomials and Scalar multiplication, where F' is subfield of F

Vector Space	$\mathbb{R}[x]$	
Subset	**Subspace**	**Dimension**
$W_1=\{p(x)/\deg(p(x))\le n\}$	**Yes**	$n+1$
$W_2=\{p(x)/p(x)$ is polynomial *of even* deg$\}$	**No**	-
$W_3=\{p(x)\,|\,p(x)$ zero polynomial or deg $(p(x))\ge k, k$ *is a* fixed natured numbe	**No**	-
$W_4=\{p(x)/p(x)$ polynomial of deg n$\}$	**No**	-

7. Which of the following is a subspace of given Vector space of $F_n[x]=\{P_n(x)\,|\,P_n(x)$, where $P_n(x)$ is a polynomials of degree at most n with coefficient from field $F\}$, then $F[x]$ is Vector Space over Field F' under addition of polynomials and Scalar multiplication, where F' is subfield of F

Vector space	$\mathbb{R}[x]$	
Subset	**Subspace**	**Dimension**
$W_5=\{p(x)\in F_n[x]/p(\alpha)=0,\alpha\in F\}$	**Yes**	n
$W_6=\{p(x)\in F_n[x]/p^k(\alpha)=0,\ \mathrm{p}^{k+1}(x)\neq$	**No**	-
$W_7=\{p(x)\in F_n[x]/p(x)=p(1-x)\}$	**yes**	$\left[\frac{n}{2}\right]+1$
$W_8=\{p(x)\in F_n[x]\,|\,p(\alpha)=p(\beta)$ $,\alpha\neq\beta\in F(\textit{fixed})\}$	**Yes**	n

$W_9 = \{p(x) \in F_n[x] / p(0) = 0\}$	**Yes**	n
$W_{10} = \{p(x) \in F_n[x] \| p(0) = 1\}$	**No**	-
$W_{11} = \{p(x) \in F_n[x] \| p(x) = p(-x)\}$	**Yes**	$\left[\frac{n}{2}\right]$
$W_{12} = \{p(x) \in F_n[x] \| p(1) \geq 0\}$	**No**	-
$W_{13} = \{p(x) \in F_n[x] \| p'(0) = 0\}$, *where* $p'(x)$ *is darivative of* $p(x)$	**Yes**	n
$W_{14} = \{p(x) \in F_n[x] \| p(x)$ has a root in interval $[-1,1]\}$	**No**	-

CHAPTER 2

LINEAR TRANSFORMATION AND ITS PROPERTIES

Definition of Linear Transformation, Range and Null Space, Kinds of Linear Transformation, Representation of Transformation by Matrices, Linear Functional, Dual Space, Dual Basis, Annihilators,

2.1. Definitions

Linear Transformation: Let $V(F)$ and $V'(F)$ be vector spaces (over F). We call a function $T:V\to V'$ a linear transformation from $V(F)$ to $V'(F)$ if, for all $x,y\in \mathrm{V}$ and $c\in F$, we have

1. $\mathrm{T}(x+y)=\mathrm{T}(x)+\mathrm{T}(y)$ and
2. $\mathrm{T}(cx)=c\mathrm{T}(x)$

Example:

$\mathbb{R}^n=\{(a_1,a_2,...,a_n)\mid a_i\in\mathbb{R}\}$ be a vector space over field $\mathbb{R}$ then

1. $T:\mathbb{R}^n\to\mathbb{R}^n$ such that $T(a_1,a_2,...,a_n)=(a_n,a_{n-1},...,a_1)$
2. $T:\mathbb{R}^n\to\mathbb{R}^n$ such that $T(a_1,a_2,...,a_n)=(a_1+a_2,a_2+a_3,...,a_n+a_1)$
3. $T:\mathbb{R}^n\to\mathbb{R}^n$ such that $T(a_1,a_2,...,a_n)=(a_1,a_2,...a_{n-1},0)$
4. $T:\mathbb{R}^n\to\mathbb{R}^n$ such that $T(a_1,a_2,...,a_n)=(0,a_1,...,a_{n-1})$

Example: $\mathbb{R}^{n\times n}=\{A=[a_{ij}]\mid a_{ij}\in\mathbb{R}\}$ be a vector space over field $\mathbb{R}$ then

1. $T:\mathbb{R}^{n\times n}\to\mathbb{R}^{n\times n}$ such that $T(A)=A^T$: where A^T denotes the transpose of A
2. $T:\mathbb{R}^{n\times n}\to\mathbb{R}^{n\times n}$ such that $T(A)=\dfrac{A+A^T}{2}$: where A^T denotes the transpose of A
3. $T:\mathbb{R}^{n\times n}\to\mathbb{R}$ such that $T(A)=Tr(A)$ where $Tr(A)$ denotes the Trace of A
4. $T:\mathbb{R}^{n\times n}\to\mathbb{R}$ such that $T(A)=\sum_{j=1}^{n}\sum_{i=1}^{n}a_{ij}$ where $A=[a_{ij}]$ denotes the Trace of A

Example: $\mathbb{R}[x] = \{p(x) \mid p(x)$ is polynomial space over field $\mathbb{R}\}$ then

1. $T : \mathbb{R}[x] \to \mathbb{R}[x]$ such that $T(p(x)) = p'(x)$: where $p'(x)$ denotes the derivative of $p(x)$

2. $T : \mathbb{R}[x] \to \mathbb{R}[x]$ such that $T(p(x)) = p''(x)$: where $p''(x)$ denotes the second derivative of $p(x)$

3. $T : \mathbb{R}[x] \to \mathbb{R}[x]$ such that $T(p(x)) = \int_0^x p(t)dt$

4. $T : \mathbb{R}[x] \to \mathbb{R}$ such that $T(p(x)) = p(0)$

5. $T : \mathbb{R}[x] \to \mathbb{R}$ such that $T(p(x)) = p'(0)$

Example: Let V be a vector space over field F

1. Identity transformation $I(x) = x$ is linear transformation from V onto V.

2. Zero transformation O defined by $O(x) = 0$ called trivial transformation

Let $V(F)$ and $V'(F)$ be vector spaces (over F), let $T : V \to V'$ be a linear transformation from $V(F)$ to $V'(F)$

Range of A Linear Transformation T : Range of T is $R(T) = \{T(x) \mid x \in V\}$

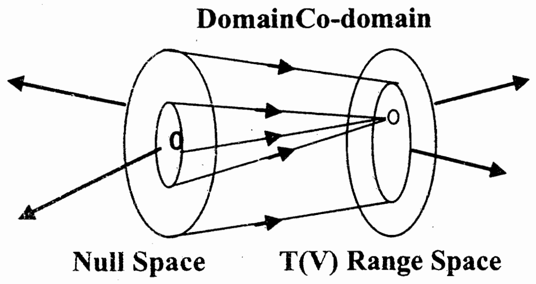
Proposition 2.1: Range of a linear transformation $T : V \to V'$ is subspace of V' and called range space of linear transformation

Rank of A Linear Transformation T : The dimension of range space of $R(T)$ of a linear transformation called Rank of a linear transformation T denoted by $\rho(T)$.

Kernel of A Linear Transformation T : kernel of T is $\ker(T) = \{x \in V \mid T(x) = 0\}$ set of all vectors which maps to zero of V'.

Proposition 2.2: kernel of a linear transformation $T : V \to V'$ is subspace of V and called null space of linear transformation T.

Nullity of a linear transformation T : The dimension of null space of $N(T)$ or $\ker(T)$ of a linear transformation called Nullity of a linear transformation T denoted by $\eta(T)$



Sylvester's Law: Let V and W be vector spaces over the field F and let T be a linear transformation from V into W .Suppose that V is finite dimensional then $\rho(T)+\eta(T)=\dim(V)$

Proposition 2.3: Let T be a linear operator on a finite dimensional vector space V . Suppose that rank $\left(T^2\right)$ = rank (T). the range and the null space of T have only zero vector in common i.e. Range $\mathrm{T} \cap \mathrm{Ker}\ \mathrm{T} = \{\mathbf{0}\}$

2.2. Kinds of Linear Transformation

Linear Operator: Let $V(F)$ be a vector space than any linear transformation from V into V is called linear operator on V .

Singular: A linear transformation called singular if there exists a nonzero vector x such that $T(x)=\mathbf{0}$ that is a linear transformation is singular if $\ker(T) \neq \mathbf{0}$ that is $\eta(T) \geq 1$

Non-Singular: A linear transformation is called non-singular if $T(x)= 0 \Rightarrow x = \mathbf{0}$ i.e, a linear transformation is **non-singular** if $\ker(T)=\{\mathbf{0}\}$

Proposition 2.4: T is non-singular $\Leftrightarrow$ T is one – one $\Leftrightarrow$ $\ker(T)=\{\mathbf{0}\}$

Invertible Linear Transformation: Let $V(F)$ and $V'(F)$ be vector spaces over the field F and let $T: V \to V'$ be a linear transformation such that T is one-one and onto then T is called invertible.

In other words, a linear transformation $T: V \to V'$ is invertible iff $\exists$ a linear transformation $S: V' \to V$ such that $TS = I = ST$

Proposition 2.5: Let V, W and Z be vector spaces over the field F . Let T be a linear transformation from V into W and S is a linear transformation from W into Z . then the composite function SoT defined by $(SoT)(x) = S(T(x))\ \forall x \in V$ is linear transformation from V into Z .

Proposition 2.6: Let T be a linear transformation from V into W , then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W .

Proposition 2.7: Let V and W be two finite dimensional vector space such that $\dim V = \dim W$ and $T: V \to W$ a linear transformation. Then the following conditions are equivalent:

1. T is invertible
2. T is non-singular
3. T is onto
4. If $\{v_1, v_2, \ldots\ldots, v_n\}$ is a basis of V , then $\{T(v_1), T(v_2), \ldots\ldots, T(v_n)\}$ is a basis of W .

Proposition 2.8: if $T: V \to W$ is a linear transformation such that dim $V = \dim W$. Then T is invertible $\Leftrightarrow$ T is $1-1$ $\Leftrightarrow$ T is non singular $\Leftrightarrow$ T is onto.

Proposition 2.9: Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. If T is invertible, then the inverse function T^{-1} is a linear transformation from W onto V

Proposition 2.10: If T and U are linear transformations on a finite dimensional vector space V such that $TU = I$, then T and U are invertible and $T^{-1} = U$.

Note: Above can be false when V is not finite dimensional.

2.3. Some the Important Linear Operators

Projection: Let $V = \mathbb{R}^2$ be a vector space over field $\mathbb{R}$ consider the function $T:\mathbb{R}^2 \to \mathbb{R}^2$

1. T is projection on $x-$axis alone $y-$ axis then $T(a,b) = (a,0)$ for all $(a,b) \in \mathbb{R}^2$ is linear operator
2. T is projection on $y-$axis alone $x-$ axis then $T(a,b) = (0,b)$ for all $(a,b) \in \mathbb{R}^2$ is linear operator
3. T is projection on $y = mx$ axis alone $y = -\frac{1}{m}x$ axis then $T(a,b) = \left(\frac{a+mb}{1+m^2}, \frac{m(a+mb)}{1+m^2}\right)$ for all $(a,b) \in \mathbb{R}^2$ is linear operator

Orthogonal Projection on Subspace: Let $V(F)$ be a vector space over field F and let W be subspace of $V(F)$, define $T:V \to V$ such that $T(w+w') = w$, where $V = W \oplus W'$, (W' is orthogonal complement subspace of W) then

1. T is a linear operator V, is called orthogonal projection on subspace W
2. $Ker(T) = W'$ and $Range(T) = W$
3. $\eta(T) = \dim W'$ and $\rho(T) = \dim W$
4. If $W \neq V$ then T neither one-one nor onto, i.e., T is singular linear operator
5. $T^2 = T$, i.e., T is idempotent linear operator

Reflection: Let $V = \mathbb{R}^2$ be a vector space over field $\mathbb{R}$ consider the function $T:\mathbb{R}^2 \to \mathbb{R}^2$

1. T is reflection about $x-$axis alone $y-$ axis then $T(a,b) = (a,-b)$ for all $(a,b) \in \mathbb{R}^2$ is linear operator
2. T is reflection about $y-$axis alone $x-$ axis then $T(a,b) = (-a,b)$ for all $(a,b) \in \mathbb{R}^2$ is linear operator
3. T is reflection about line $y = mx$, alone $y = -\frac{1}{m}x$ then $T(a,b) = \left(2\left(\frac{a+mb}{1+m^2}\right) - a, 2\left(\frac{m(a+mb)}{1+m^2}\right) - b\right)$ for all $(a,b) \in \mathbb{R}^2$ is linear operator
4. T is reflection about origin then $T(a,b) = (-a,-b)$ for all $(a,b) \in \mathbb{R}^2$ is linear Operator

Reflection on Subspace Along its Orthogonal Complement: Let $V(F)$ be a vector space over field F and let W be subspace of $V(F)$, define

$T:V\to V$ such that $T(w+w')=w-w'$, where $V=W\oplus W'$, (W' is orthogonal complement subspace of W) then

1. T is a linear operator V ,is called reflection about subspace W alone W' is orthogonal complement subspace of W
2. $Ker(T)=\{0\}$ and $Range(T)=V$
3. $\eta(T)=0$ and $\rho(T)=\dim V$
4. T one-one and onto , i.e., T is Invertible linear operator
5. $T^2=I$,i.e., T is Involuntary linear operator

Rotation: Let $V=\mathbb{R}^2$ be a vector space over field $\mathbb{R}$ consider the function $T:\mathbb{R}^2\to\mathbb{R}^2$, T is rotation in anti-clock wise by an angle θ then $T(a,b)=(a\cos\theta-b\sin\theta, a\sin\theta+b\cos\theta)$ for all $(a,b)\in\mathbb{R}^2$ is a linear operator .

1. $Ker(T)=\{0\}$ and $Range(T)=V$
2. $\eta(T)=0$ and $\rho(T)=\dim V$
3. T one-one and onto , i.e., T is Invertible linear operator
4. If T_θ & T_φ are rotation in anti-clock wise by an angle θ and φ then $T_\theta\circ T_\varphi$ rotation in anti-clock wise by an angle $\theta+\varphi$
5. $G=\Big\{T_\theta:\mathbb{R}^2\to\mathbb{R}^2 \mid \binom{n}{\theta} n=360\ \theta T_\theta\circ T_\theta\circ\ldots\circ T_\theta=T_{n\theta}$ are rotation by an angle $\theta\Big\}$ then under Composition of function G be infinite order abelian group
6. G are rotation by an angle θ then $T_\theta^n=T_\theta\circ T_\theta\circ\ldots\circ T_\theta=T_{n\theta}$
7. G are rotation by an angle θ then there exists $T_\theta^n=I$ for some $n\in\mathbb{N}$ iff θ is rational multiple of π
8. If θ is rational multiple of π and G are rotation by an angle θ such that $T_\theta^n=I$ then $\theta=k\left(\dfrac{360^\circ}{n}\right)$

Let $V=\mathbb{R}^3$ be a vector space over field $\mathbb{R}$ consider the function $T:\mathbb{R}^3\to\mathbb{R}^3$,

1. T is rotation in anti-clock wise by an angle θ in $xy-$plane then $T(a,b,c)=(a\cos\theta-b\sin\theta, a\sin\theta+b\cos\theta, c)$ for all $(a,b,c)\in\mathbb{R}^3$ is a linear operator .
2. T is rotation in anti-clock wise by an angle θ in $yz-$plane then $T(a,b,c)=(a, b\cos\theta-c\sin\theta, b\sin\theta+c\cos\theta)$ for all $(a,b,c)\in\mathbb{R}^3$ is a linear operator .
3. T is rotation in anti-clock wise by an angle θ in $xz-$plane then $T(a,b,c)=(a\cos\theta-c\sin\theta, b, a\sin\theta+c\cos\theta)$ for all $(a,b,c)\in\mathbb{R}^3$ is a linear operator .

Let $V=\mathbb{R}[x]$ be a polynomial space over field $\mathbb{R}$ consider the linear transformation $T:\mathbb{R}[x]\to\mathbb{R}[x]$

1. $T(p(x))=\dfrac{d}{dx}(p(x))$: for all $p(x)$ is polynomial

2. $T(p(x))=\int\limits_0^x p(t)dt$: for all $p(x)$ is polynomial

3. Let $V=\mathbb{R}^{n\times n}$ be a vector space of real matrices over field $\mathbb{R}$ and let $A\in\mathbb{R}^{n\times n}$ define then $T_A:V\to V$ such that $T_A(X)=AX$ or $T_A(X)=XA$ for all $X\in\mathbb{R}^{n\times n}$ both are linear transformations

4. Let $V=\mathbb{R}^n$ be a vector space of real matrices over field $\mathbb{R}$ and let $A\in\mathbb{R}^{n\times n}$ define then $T_A:V\to V$ such that $T_A(X)=AX$ for all $X=(x_1,x_1,...,x_n)^T\in\mathbb{R}^n$ is linear transformations

Invariant Subspace: Let V be vector space, and let $T:V\to V$ be linear. Then a subspace W of V is said to be T-Invariant if $T(x)\in W$, for every $x\in W$, that is $T(W)\subseteq W$

Induced Linear Operator: Let V be vector space and $T:V\to V$ be linear operator.

Proposition 2.11: If W is T-invariant subspace and then the restriction $T_w:W\to W$ of T on W is linear operator on W.

Proposition 2.12: Let V be vector space, and let $T:V\to V$ be linear operator then

1. The subspaces $\{0\},V,R(T)$, and $N(T)$ are all T-invariant
2. If T is the projection on W along some subspace W' then W and W' is T-invariant and that $T_w=I_w$
3. If W is T-invariant. Then $N(T_w)=N(T)\cap W$ and $R(T_w)=T(W)$

Linear Operator Induced by Invariant Subspace: Let W be a T-Invariant subspace of V then define $\hat{T}$ restriction of T on W that is $\hat{T}:W\to W$ such that $\hat{T}(x)=T(x),\forall x\in W$ then $\hat{T}$ is a linear operator on W called linear operator induced by T on W.

Linear Operator Induced by Quotient Space Subspace: Let W be a T-Invariant subspace of V then define $T':\dfrac{V}{W}\to\dfrac{V}{W}$ such that $T'(x+W)=T(x)+W,\forall x+W\in\dfrac{V}{W}$ then T' is a linear operator on $\dfrac{V}{W}$ called linear operator induced by T on $\dfrac{V}{W}$.

2.4. Properties on LinearTransformations

Let $V(F)$ and $V'(F)$ be vector spaces (over F). We call a function $T:V\to V'$ a linear transformation from $V(F)$ to $V'(F)$

1. If T is linear, then $T(0)=0$.
2. T is linear if and only if $T(cx+y)=cT(x)+T(y)$ for all $x,y\in V$ and $c\in F$.
3. If T is linear, then $T(x-y)=T(x)-T(y)$ for all $x,y\in V$.
4. T is linear if and only if, for $x_1,x_2,\ldots,x_n\in V$ and $a_1,a_2,\ldots,a_n\in F$, we have $T\left(\sum_{i=1}^{n}a_ix_i\right)=\sum_{i=1}^{n}a_iT(x_i)$
5. Let V and W be vector spaces, and let $T:V\to W$ be linear. If $\beta=\{v_1,v_2,\ldots,v_n\}$ is a basis for V, then $R(T)=\text{span}(T(\beta))=span(\{T(v_1),T(v_2),\ldots,T(v_n)\})$

   How to Apply Above: Let $V=\{p(x)\mid$ where $p(x)$ polynomial with degree at-most 2 with real coefficient is$\}$ and $W=\{A\mid A$ be a 2×2 real matrix$\}$ then $T:V\to W$ such that $T(p(x))=\begin{pmatrix}p(1) & p(0)\\ 0 & p(0)-p(1)\end{pmatrix}$ be a linear transformation then $\rho(T)$ and $\eta(T)$ is

   Consider standard basis $\beta=\{1,x,x^2\}$ then $R(T)=span(T(\beta))$ then rank of T is number of linearly independent vectors in $T(\beta)$.

   $T(1)=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, T(x)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $T(x^2)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ ∴ there are only two linearly independent vector in $T(\beta)$ hence $\rho(T)=2$ and $\eta(T)=1$ implies T is neither one-one nor onto.
6. Let V be vector space, and let $T:V\to V$ be linear operator. If V is finite-dimensional, the $\text{nullity}(T)-rank(T)=\dim(V)$ implies T is bijective.
7. Let V and W be vector spaces, and let $T:V\to W$ be linear. Then T is one-to-one if and only if $N(T)=\{0\}$
8. Let V and W be vector spaces over F. If $\{v_1,v_2,\ldots,v_n\}$ is a basis of V then For any $w_1,w_2,\ldots,w_n$ in W, there exists exactly one linear transformation $T:V\to W$ such that $T(v_i)=w_i$ for $i=1,2,\ldots,n$
9. Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1,v_2,\ldots v_n\}$. If $U,T:V\to W$ are linear and $U(v_i)=T(v_i)$ for $i=1,2,\ldots,n$, then $U=T$ that if two linear transformation are agree on basis then they are identical.
10. Let V and W be vector spaces and $T:V\to W$ be linear transformation then
    1. If $x_1,x_2,\ldots,x_k$ are linearly independent vectors in V then $T(x_1),T(x_2),\ldots,T(x_k)$ may or may not linearly independent in W.
    2. If $x_1,x_2,\ldots,x_k$ are linearly dependent vectors in V then $T(x_1),T(x_2),\ldots,T(x_k)$ are linearly dependent is W.

3. If $T(x_1), T(x_2), ..., T(x_k)$ are linearly dependent in W then $x_1, x_2, ..., x_k$ are may or may not linearly dependent vectors in V.
4. If $T(x_1), T(x_2), ..., T(x_k)$ are linearly independent in W then $x_1, x_2, ..., x_k$ are linearly independent vectors in V.

11. Let V and W be vector spaces and $T : V \to W$ be a one-one linear transformation then

1. If $x_1, x_2, ..., x_k$ are linearly independent vectors in V then $T(x_1), T(x_2), ..., T(x_k)$ iff linearly independent in W.
2. If $x_1, x_2, ..., x_k$ are linearly dependent vectors in V iff $T(x_1), T(x_2), ..., T(x_k)$ are linearly dependent is W.
3. T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.
4. $\beta = \{v_1, v_2, ... v_n\}$ is a basis for V and T is one-to-one and onto then $T(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for W

12. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.

1. if $\dim(V) < \dim(W)$, then T cannot be onto.
2. if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

13. Let V and W be vector spaces over F. then set of all linear transformations form V into W by $L(V, W)$. In the case that $V = W$, we write $L(V)$ instead of $L(V, W)$

14. Let $V = \mathbb{R}[x]$, and for $j \geq 1$ define $T_j(p(x)) = p^{(j)}(x)$, where $p^{(j)}(x)$ is the j^{th} derivative of $p(x)$ .then the set $\{T_1, T_2, ..., T_n\}$ is a linearly independent subset of $L(V)$ for any positive integer n

15. Let V be a vector space. Let $T, U_1, U_2 \in L(V)$ (Vector space of all linear operators on V) Then

1. $T(U_1 + U_2) = TU_1 + TU_2$ and $(U_1 + U_2)T = U_1T + U_2T$
2. $T(U_1U_2) = (TU_1)U_2$
3. $TI = IT = T$
4. $\alpha(U_1U_2) = (aU_1)U_2 = U_1$ for all scalars a

16. Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear

1. If $rank(T) = rank(T^2)$, then $R(T) \cap N(T) = \{0\}$ and $V = R(T) \oplus N(T)$
2. $V = R(T^k) \oplus N(T^k)$ for some positive integer k

17. Let T be an invertible linear transformation form V to W. Then V is finite-dimensional if and only if W is finite-dimensional and $\dim(V) = \dim(W)$

Ordered Bases: If V is a finite dimensional vector space, then an ***ordered basis*** for V is a finite sequence of vectors which is linearly independent and spans V.

i.e. let $S=\{x_1, x_2, \ldots\ldots\ldots, x_n\}$ be a basis of $V(F)$ then it is called an ordered basis if the position of vectors in S is fixed i.e. the set S is made ordered, hence if $S=\{x_1, x_2, x_3\}$ is an ordered basis then the basis $\{x_2, x_1, x_3\}$ is considered as different ordered basis.

Coordinates: Let $B=\{e_1, \ldots\ldots, e_n\}$ be a basis of an n-dimensional vector space V over a field K, and let x be any vector in V. Since $\{e_i\}$ generates V, x can be written as linear combination of the e_i. i.e.

$$x = a_1e_1 + a_2e_2 + \ldots\ldots + a_ne_n,\ a_i \in K$$

Since the e_i's are independent, such a representation is unique i.e. the n scalars $a_1, \ldots, a_n$ are completely determined by the vector x and the basis $\{e_1, \ldots\ldots, e_n\}$. We call these scalars the coordinates of x in $\{e_1, \ldots\ldots, e_n\}$, and we call the transpose of n-tuple $(a_1, \ldots\ldots, a_n)$ the coordinate vector of x relative to $\{e_1, \ldots\ldots, e_n\}$ and denote it by $[x]_B$ or simply $[x]$

Hence $[x]_B=(a_1, a_2, \ldots\ldots\ldots, a_n)^t$

Matrix Representation of Linear Transformation: Let T be a Linear Transformation from vector space $V(F)$ into $V'(F)$. Let $\beta=\{x_1, x_2, \ldots, x_n\}$ be a ordered basis of V and $\beta'=\{y_1, y_2, \ldots, y_m\}$ ordered basis of $V'(F)$.

Then As $T(x_i)\in V'$ each $T(x_i)$ is a linear combination of the elements of the basis $\beta'=\{y_1, y_2, \ldots, y_m\}$.

$$T(x_i)=\sum_{j=1}^{m} a_{ij}y_j = a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{im}y_m,\ \forall i=1,2,\ldots,n \text{ That is}$$

$$T(x_1)=\sum_{j=1}^{m} a_{1j}y_j = a_{11}y_1 + a_{12}y_2 + \ldots + a_{1m}y_m$$

$$T(x_2)=\sum_{j=1}^{m} a_{2j}y_j = a_{21}y_1 + a_{22}y_2 + \ldots + a_{2m}y_m$$

$$T(x_n)=\sum_{j=1}^{m} a_{nj}y_j = a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nm}y_m$$

The matrix of coefficients is $\begin{bmatrix} a_{11} & a_{12}--- & a_{1n} \\ a_{21} & a_{22}--- & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2}--- & a_{nn} \end{bmatrix}$

The transpose of the above matrix of coefficients is called the matrix representation of T with respect to β and β' and denoted by $[T:\beta,\beta']$.

$$[T:\beta,\beta']=\begin{bmatrix} a_{11} & a_{21}--- & a_{n1} \\ a_{12} & a_{22}--- & a_{n2} \\ \vdots & \vdots & \vdots \\ a_{1n} & a_{2n}--- & a_{nn} \end{bmatrix}$$

Example:

1. Consider the following bases of R^2 $\{e_1 = (1, 0), e_2 = (0, 1)\}$ and $\{e'_1 = (1, 3), e'_2 = (2, 5)\}$ Determine the matrix $[T]_{B'}$:

$$e'_1 = (1, 3) = (1, 0) + 3(0, 1) \quad ,$$

$$e'_2 = (2, 5) = 2(1, 0) + 5(0, 1) \quad ,$$

$$i.e. e'_1 = e_1 + 3e_2 \quad \& \quad e'_2 = 2e_1 + 3e_2$$

$$\therefore \text{ Transition matrix } \quad [T]_{B'} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Find the matrix of linear operator D on the vector space P of polynomials of degree less than equal to n where $D(P(x)) = P'(x)$

Solution: Take $\beta = \{1, x, x^2, \text{-----} \ x^n\}$ be a basis of vector space P.Now

$$D(1) = 0, = 0.1 + 0.x + \text{------} + 0.x^n$$

$$D(x) = 1 = 1.1 + 0.x + \text{----------} + 0.x^n$$

$$D(x^2) = 2x = 0.1 + 2.x + \text{-----} + 0.x^n$$

$$D(x^n) = nx^{n-1} = 0.1 + 0.x + \text{-----} + n.x^{n-1} + 0.x^n$$

$$\therefore \ D = \begin{bmatrix} 0 & 1 & 0\text{---}0 & 0 \\ 0 & 0 & 2...0 & 0 \\ : & : & : \quad : & : \\ : & : & :\text{---} : & : \\ : & : & :\text{---} : & n \\ 0 & 0 & 0\text{---}0 & 0 \end{bmatrix}$$

Linear Functional: Let $V(F)$ be a vector space over a field F. Then, a mapping f from V to F (i.e., $f : V \to F$) is called a linear functional on V if $f(ax + by) = af(x) + bf(y), \ \forall x, y \in V$ and $a, b \in F$.

Thus, for $x \in V$,$f(x) \in F$ and this $f(x)$ being a scalar, the linear functional f on V is a scalar valued function. Since F, the field of scalars, is also a vector space over F. We shall be using the same symbol F both for the field as well as for vector space over itself.

If f is a linear functional on $V(F)$, then f is a L.T. from V to F. Therefore, f possesses all the properties of linear transformations also. Thus

1. $f(0) = 0$ where 0 is the null vector of V and 0 is the zero element of F.
2. $f(-x) = -f(x)$, $\forall x \in V$

Example of Linear Functional:

1. **Zero Functional:** A mapping $f: V(F) \to F$ defined by $f(x) = 0, \forall x \in V$, is a linear functional and is denoted by $\hat{0}$.
2. **Negative of a Linear Functional:** Let $f: V(F) \to F$ be a linear functional. Then, the mapping $-f$ defined by: $-f : V(F) \to F$ by $(-f)(x) = -f(x), \forall x \in V$ is a linear functional on V

Dual Space: The set V^* of all linear functional f defined on a vector space $V(F)$ is a vector space over F with addition and scalar multiplication defined by: $(f_1 + f_2)(x) = f_1(x) + f_2(x), \forall x \in V$ and $(af_1)(x) = af_1(x), a \in F$ where f_1, f_2 are linear functional on V.

This vector space V^*(also denoted by V' or $\hat{V}$) is called the dual space or conjugate space or algebraic dual or the algebraic conjugate of V over F.

Proposition 2.13: Let $V(F)$ be an n-dimensional vector space over F and $B = \{x_1, x_2, \ldots, x_n\}$ be an orderd basis of V. Then, for any ordered set $S = \{a_1, a_2, \ldots\ldots\ldots, a_n\}$ of n scalars, there exists a unique linear functional f on V such that $f(x_i) = a_i, i = 1, 2, \ldots\ldots, n$.

Important Results:

1. If f is a linear functional on a finite dimensional vector space V, then f is completely determined if we mention the images of the elements of a basis set of V under f.
2. Two linear functional on V are equal iff they agree on a basis set of V.
3. If $V(F)$ is an n-dimensional vector space and $B = \{x_1, x_2, \ldots\ldots\ldots, x_n\}$ is a basis for V, then there is a uniquely determined basis $B^* = \{f_1, f_2, \ldots\ldots\ldots, f_n\}$ for V^* such that

   $f_i(v_j) = \delta_{ij}$, where δ_{ij} is the Kronecker Delta $\delta_{ij} = \begin{cases} 0, & if\ i \neq j \\ 1, & if\ i = j \end{cases}$

   i.e. The dual space V^* of an n-dimensional vector space $V(F)$ is n-dimensional. The basis $B^* = \{f_1, f_2, \ldots\ldots, f_n\}$ of $V^*(F)$ is called the dual of the basis B. Moreover $V^*(F) \cong V(F)$
4. Let $V(F)$ be an n-dimensional vector space Let $B = \{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for V & Let $B^* = \{f_1, f_2, \ldots\ldots\ldots, f_n\}$ be the dual basis of B. Then,

   a. Each linear functional f on V can be written as : $f = \sum_{i=1}^{n} f(v_i) f_i$

   and

   b. Each $v \in V$, can be written as : $v = \sum_{i=1}^{n} f_i(v) v_i$

Bidual Space or Second Dual Space: We know that every vector space $V(F)$ possesses a dual space $V^*(F)$ consisting of all linear functional on V. Now, $V^*(F)$ being a vector space possesses a dual space $(V^*)^*$ consisting of all linear functionals on V^*. This dual space $(V^*)^*$ of V^* is called the second dual space of V and is generally denoted by V^{**}. It consists of all linear functionals on vector space V^*. Also, dim V = dim. V^* = dim. V^{**}, if V is finite dimensional Hence, $V \cong V^* \cong V^{**}$

Let $V(F)$ be a finite dimensional vector space. If x is any vector in V, the mapping f_x from V^* to F defined by $f_x(g) = g(x), \forall g \in V^*$ is a linear functional on V^*, *i.e.*, $f_x \in V^{**}$. Also, the mapping $x \to f_x$ is an isomorphism of V onto V^{**}. This isomorphism is called the natural isomorphism

The mapping $x \to f_x$ is not onto if V is not finite dimensional although it is always one-one and linear.

1. If $V(F)$ is a finite dimensional vector space and f^* is a linear functional on the dual space V^* of V, then there is a unique vector $x \in V$ such that $f^*(g) = g(x), \forall g \in V^*$

   Reflexivity: If $V(F)$ is a vector space and V^{**} is its second dual space, then the property that the natural corresponding $x \to f_x$ between V and V^{**} is an isomorphism is called reflexivity between V and V^{**}. The two vector spaces V and V^{**} are called reflexive.

Annihilator of a Subspace: Let W be a subspace of a vector space $V(F)$. The set of all linear functionals $f \in V^*$ such that $f(w) = 0$ for every $w \in W$, is denoted by W° and is called an annihilator of the subspace W. i.e. $W^{o} = \{f \mid f \in V^* \text{ and } f(w) = 0 \text{ for } w \in W\}$

Note:

1. $V^{o} = \{\hat{0}\}$, where $\hat{0}$ is the zero functional
2. $\{0\}^{o} = V^*$
3. If W is a subspace of a vector space $V(F)$ then W° (the annihilator of W) is a subspace of V^*.
4. If $V(F)$ is a finite dimensional vector space and W is a subspace of it, then dim W + dimW^{o} = dimV.

Examples:

1. If $\{\ v_1 = (1,-2,3),\ v_2 = (1,-1,1),\ v_3 = (2,-4,7)\ \}$ is a basis of $\mathbb{R}^3$, then find the dual basis of $\{v_1, v_2,\ v_3\}$.

Solution: $B = \{\ v_1,\ v_2,\ v_3\}$ is a basis of $\mathbb{R}^3$. Let $B^* = \{f_1, f_2, f_3\}$ be the dual basis of B. Then $f_i(v_j) = \delta_{ij} = \begin{cases} 1 & if\ \ i = j \\ 0 & if\ \ i \neq j \end{cases}$

Let $f_1(x, y, z) = a_1x + a_2y + a_3z$,

$f_2(x, y, z) = b_1x + b_2y + b_3z$.

$f_3(x, y, z) = c_1x + c_2y + c_3z$,

$f_1(v_1) = f_1(1, -2, 3) = a_1 - 2a_2 + 3a_3 = 1$

$f_1(v_2) = f_1(1, -1, 1) = a_1 - a_2 + a_3 = 0$,

$f_1(v_3) = f_1(2, -4,7) = 2a_1 - 4a_2 + 7a_3 = 0$

Solving these equations, we get $a_1 = -3,\ a_2 = -5,\ a_3 = -2$

$\therefore f_1(x, y, z) = -3x - 5y - 2z$

Let us find f_2, $f_2(v_1) = f_2(1, -2, 3) = b_1 - 2b_2 + 3b_3 = 0$

$f_2(v_2) = f_2(1, -1, 1) = b_1 - b_2 + b_3 = 1$

$f_2(v_3) = f_2(\ 2, -4,\ 7) = 2b_1 - 4b_2 + 7b_3 = 0$

Solving these equations, we get $b_1 = 2,\ b_2 = 1, b_3 = 0$

$\therefore \qquad f_2(x, y, z) = 2x + y + 0z = 2x + y$

Now find f_3, $f_3(v_1) = f_3(1, -2, 3) = c_1 - 2c_2 + 3c_3 = 0$

$f_3(v_2) = f_3(1, -1, 1) = c_1 - c_2 + c_3 = 0$

$f_3(v_3) = f_3(2, -4, 7) = 2c_1 - 4c_2 + 7c_3 = 1$

Solving these equations, we get $C_1 = 1,\ C_2 = 2,\ C_3 = 1$

$\therefore \qquad f_3(x, y, z) = x + 2y + z$

Here dual basis B* is $\{f_1, f_2, f_3\}$; Where f_1, f_2, f_3 are defined by (1), (2) and (3).

2. Let V be the vector space of all polynomials of degree ≤ 2 over the field $\mathbb{R}$. If ϕ_1, ϕ_2, ϕ_3 are linear functionals on V given by:

$\phi_1(f(t)) = \int_0^1 f(t)\,dt$, $\phi_2(f(t)) = f'(t)$, $\phi_3(f(t)) = f(0)$, where

$f(t) = a + bt + ct^2 \in V$, and $f'(t) = \frac{d}{dt}f(t)$, Find the basis $\{f_1(t), f_2(t), f_3(t)\}$ of V which is dual to $\{\phi_1, \phi_2, \phi_3\}$

2.5. System Linear Equations

Let $A=[a_{ij}]$ be $m\times n$ matrix with entries from field $a_{ij}\in F$, then we can assume T is a linear transformation $T:F^n\rightarrow F^m$ whose matrix representation with respect to some ordered basis β is A.

As T be a linear transformation from F^n to F^m so for each $X\in F^n$ implies $T(X)\in F^m$ that is we can define a system $T(X)=b$ which is equivalent to system $AX=b$ where $X=(x_1,x_2,...x_n)\in F^n$ and $b=(b_1,b_2,...,b_m)\in F^m$

called system of m -equations and in n -variables.

Solution of System of Equation: Consider the System of Equation $AX=b$ of m -equations and in n -variables.

Then solution of $AX=b$ we mean for given matrix (or linear transformation) A and column vector $b\in F^m$ there exist a column vector $X=(x_1,x_2,...x_n)\in F^n$ such that $AX=b$.

Consistent System of Equation: The System of Equation $AX=b$ of m -equations and in n -variables called consistent if it has solution

Inconsistent System of Equation: The System of Equation $AX=b$ of m -equations and in n -variables called inconsistent if it does not have solution.

Homogenous and Non-Homogenous System of equations: The System of Equation $AX=b$ of m -equations and in n -variables called homogenous if $b=0$ otherwise non- homogenous system of equations.

Necessary and Sufficient Condition for System of Equation to be Consistent: Consider the System of Equation $AX=b$ of m -equations and in n -variables. Where $X=(x_1,x_2,...x_n)\in F^n$ and $b=(b_1,b_2,...,b_m)\in F^m$ and let $S=\{c_1,c_2,...,c_n\}$ denotes the columns of matrix A.

1. System of Equation has a solution iff $\exists\ X=(x_1,x_2,...x_n)\in F^n$ such that $AX=b$
2. $AX=b$ has a solution iff $X\in F^n$ in range set of A that is $X\in R(A)$
3. $AX=b$ has a solution iff $X\in F^n$ is in column space of A
4. $AX=b$ has a solution iff $X\in F^n$ is in spanning set of $S=\{c_1,c_2,...,c_n\}$
5. $AX=b$ has a solution iff $X\in F^n$ is linearly dependent on the set $S=\{c_1,c_2,...,c_n\}$
6. $AX=b$ has a solution iff number of linearly independent vectors in $S=\{c_1,c_2,...,c_n\}$ and number of linearly independent vectors in $S_1=S\cup\{X\}$ are same
7. $AX=b$ has a solution iff dimension of $span(S)$ and dimension of $Span(S_1)$, where $S_1=S\cup\{X\}$
8. $AX=b$ has a solution iff $\rho(A)=\rho(A:b)$, where $A:b$ is augmented matrix

Analysis of Solution of the Homogenous System of Equations: Consider the homogenous system of equation $AX = 0$ of m -equations and in n -variables. $A = [a_{ij}]$ be $m \times n$ matrix with entries from infinite field $a_{ij} \in F$,

1. AS $0 \in R(A)$ implies system is always consistent hence always has trivial solution
2. $X \in F^n$ is a solution iff $X \in \ker(A)$
3. System has non-trivial solution iff $\ker(A) \neq \{0\}$
4. System has non-trivial solution iff A is not one-one
5. System has non-trivial solution iff $\rho(A) < n$
6. System has non-trivial solution iff $\eta(A) > 0$
7. If number of equations (m) < number of variables (n) then system always has nontrivial solution
8. System has non-trivial solution iff it has infinite many solution

Existence of Solution of the Non-Homogenous System of Equations: Consider the non-homogenous system of equation $AX = b$ of m -equations and in n -variables. $A = [a_{ij}]$ be $m \times n$ matrix with entries from infinite field $a_{ij} \in F$,

1. "Necessary and Sufficient condition for system of equation to be consistent is $b \in R(A)$ "
2. If the non-homogenous system of equations consistent. If w be the solution of $AX = b$ then
   - **a.** $w \neq 0$
   - **b.** The set $w + \ker(A)$ coset of kernel represents the set of all solution of the system
   - **c.** If w be the solution of $AX = b$ then every solution y of $AX = b$ is member of $w + \ker(A)$ and vice versa.
   - **d.** $W = Span(w + \ker(A))$ is vector space generated by solution space of non-homogenous System of equations
   - **e.** $\dim(W) = n - \rho(A) + 1$

Analysis of Solution of the Non-Homogenous System of Equations: Consider the non-homogenous system of equation $AX = b$ of m -equations and in n -variables. $A = [a_{ij}]$ be $m \times n$ matrix with entries from infinite field $a_{ij} \in F$,

1. If $\rho(A) = m$ or all the column of A are linearly independent or Transformation $T : F^n \to F^m$ is onto then system of equation always has solution and $m \leq n$
   - **a.** If $m = n$ implies $\eta(A) = 0$ implies system has non-trivial unique solution hence dimension of $\dim(W) = 1$
   - **b.** If $m < n$ implies $\eta(A) = n - m > 0$ implies system has infinite many solution and dimension of $\dim(W) = n - m + 1$

2. If $\rho(A)=n$ or all the rows of A are linearly independent or Transformation $T:F^n \to F^m$ is one-one then system of equation has at most one solution and $m \geq n$

  a. If $m=n$ implies $\rho(A)=m$ implies system has non-trivial unique solution hence dimension of $\dim(W)=1$

  b. If $m>n$ implies $\rho(A)<m$ implies system has either no- solution if $b \notin R(A)$ or has unique solution if $b \in R(A)$

3. If $\rho(A)=r, r \neq n,m$ or neither all the rows of A nor all columns of a are linearly independent or Transformation $T:F^n \to F^m$ is neither one-one nor onto then the system has either no- solution if $b \notin R(A)$ or infinite solution if $b \in R(A)$

Linear Transformation:

- Linear Transformation and its properties ,
- Range space and Null Space of linear transformation , Sylvester's law
- Kinds of Linear transformation singular , Non-singular ,linear operators,
- Representation of Transformation by Matrices
- Linear Functional, Dual Space, Dual Basis, Annihilators,

CHAPTER 3

MATRICES AND THEIR PROPERTIES

3.1. Definitions

Let $A_{m\times n} = [a_{ij}]$ be matrix with either real entries or complex entries

Equal Matrices: Two matrices are said to be equal if they have the same order and each element of one is equal to the corresponding element of the other

Row Matrix: A matrix having a single row is called a row matrix. Also called row vector .

Column Matrix: A matrix having a single column is called a column matrix. Also called column vector .

Square Matrix: An $m \times n$ matrix A is said to be a square matrix if $m = n$ i.e. number of rows = number of columns.

Note: In a square matrix the diagonal from left hand side upper corner to right hand side lower corner is known as leading diagonal or principal diagonal.

Traces of a Matrix: The sum of the elements of a square matrix A lying along the principal diagonal is called the trace of A denoted by $Tr(A)$.

Diagonal Matrix: A square matrix all of whose elements except those in the leading diagonal, are zero is called a diagonal matrix.

Example:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ also denoted by } A = diag(3,5,-1)$$

Scalar Matrix: A diagonal matrix whose all the elements are equal is called a scalar matrix.

Unit Matrix or Identity Matrix: A diagonal matrix of order n which has unity for all its elements, is called a unit matrix of order n and is denoted by I_n.

Triangular Matrix: A square matrix in which all the elements below the principal diagonal are zero is called Upper Triangular matrix and a square matrix in which all the elements above the principal diagonal are zero is called Lower Triangular matrix.

Note:

1. Diagonal matrix is both upper and lower triangular
2. A triangular matrix A = $[a_{ij}]_{n\times n}$ is called strictly triangular if $a_{ii} = 0$ for $1 \le i \le n$.

Null Matrix: If all the elements of a matrix (square or rectangular) are zero, it is called a null or zero matrix.

Transpose of a Matrix: The matrix obtained from any given matrix A, by interchanging its rows and columns, is called the transpose of A and is denoted by A^T.

Conjugate of a Matrix: The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\bar{A}$

Transpose conjugate of a Matrix: The transpose of the conjugate of a matrix is called transposed conjugate of A and is denoted by A^θ. The conjugate of the transpose of A is the same as the transpose of the conjugate of A

3.2. Special Matrices

Symmetric Matrices: A square matrix $A=[a_{ij}]$ is said to be symmetric when $A^T = A$.

Skew Symmetric Matrices: A square matrix A = [a_{ij}] is said to be Skew Symmetric Matrices if $A^T = -A$

Note: All the leading diagonal elements are zero.

Hermitian Matrices: A square matrix $A=[a_{ij}]$ is said to be **Hermitian if** $A^\theta = A$

Skew - Hermitian Matrices: A square matrix $A=[a_{ij}]$ is said to be skew **Hermitian if** $A^\theta = -A$

Singular and Non-singular Matrices: Any square matrix A is said to be non-singular if |A| ≠ 0, and a square matrix A is said to be singular if |A| = 0. Here |A| (or det(A) or simply det A) means corresponding determinants of square matrix A

Unitary Matrix: A square matrix is said to be unitary if $AA^\theta = I = A^\theta A$

Orthogonal Matrix: A square matrix is said to be unitary if $AA^T = I = A^T A$

Idempotent Matrix or Projection Matrix: A square matrix A is called idempotent provided it satisfies the relation $A^2 = A$

Involuntary Matrix or reflection Matrix; A square matrix A is called **Involuntary** provided it satisfies the relation $A^2 = I$

Nilpotent Matrix: A square matrix A is called a nilpotent matrix if there exists a positive integer k such that $A^k = 0$ and such least positive integer k such that $A^k = 0$ is called the index of the nilpotent matrix A.

Bi-diagonal Matrix: A matrix with elements only on the main diagonal and either the super-diagonal or sub-diagonal.

Block Matrix: A matrix partitioned in sub-matrices called blocks.

Block-diagonal Matrix: A block matrix with entries only on the diagonal.

Boolean Matrix: A matrix whose entries are all either 0 or 1.

Permutation Matrix: A matrix representation of a permutation, a square matrix with exactly one 1 in each row and column, and all other elements 0.

Vandermonde Matrix: a matrix with the terms of a geometric progression in each row, i.e., an m × n matrix

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{n-1} \end{bmatrix}$$

The determinant of a square Vandermonde matrix:

$$\det(V) = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i).$$

Companion Matrix: The square matrix defined as

$$C(p) = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix}.$$

And the monic polynomial denotes the characteristics equation

$$p(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1} + t^n$$

3.3. Properties on Matrices

1. $A + B = B + A$ $[Commutative]$
2. $(A + B) + C = A + (B + C)$ $[Associativity]$
3. $A + O = A = O + A$ $[Existence\ of\ Identity]$
4. $-A + A = O = A - A$ $[Existence of Inverse]$
5. $A + B = A + C \Rightarrow B = C \Rightarrow B + A = C + A$

   $\Rightarrow B = C$ $[Concellation law]$
6. The equation $A + X = 0$ has a unique solution in the set of all $A(m \times n)$ matrices
7. $k(A + B) = kA + kB$ $[k\ be\ any\ scalar]$
8. $(p + q)A = pA + qA [p, q\ are\ two\ scalar]$
9. $(pq)A = p(qA)$
10. $(-K)A = -(KA) = K(-A)$
11. $1A = A$, $(-1)A = -A$

Put Your Own Notes

12. $-(A+B) = -A-B$

13. If the product AB exists, then it is not necessary that BA Product also exist.

14. $A(BC) = (AB)C$ $[Associativity]$

15. $A(B+C) = AB + AC[distributive]$

16. The equation $AB = 0$ does not necessary Imply that at least one of matrices A and B must be a zero matrix.

17. If $AB = 0$, then it does not necessarily imply that $BA = 0$

18. $AI_n = A = I_n A$

19. $A^2 - B^2 = (A-B)(A+B)$ is true only if matrices A and B commute

20. The Product of two triangular matrices is itself triangular.

21. If a diagonal matrix is commutative with every matrix of the same order then it is necessary a scalar matrix.

22. If $AB = A$ and $BA = B$ then A and B are idempotent

23. If B is an idempotent matrix, then $A = I - B$ is also idempotent and that $AB = BA = 0$

24. A is involuntary iff $(I+A)(I-A) = 0$

25. $tr(kA) = k.tr(A)$

26. $tr(A+B) = tr(A) + tr(B)$

27. $tr(AB) = tr(BA)$

28. $(A^T)^T = A$

29. $(A+B)^T = A^T + B^T$

30. $(AB)^T = B^T A^T$

31. Let $\overline{A}$ denotes the Complex conjugate of A then

    a. $(\overline{A}) = A$

    b. $\overline{(A+B)} = \overline{A} + \overline{B}$

    c. $\overline{(KA)} = \overline{K}(\overline{A})$

    d. $(A^{\theta})^{\theta} = A$

    e. $(A+B)^{\theta} = A^{\theta} + B^{\theta}$

    f. $(kA)^{\theta} = \overline{k}A^{\theta}$

    g. $(AB)^{\theta} = B^{\theta}A^{\theta}$

    h. $tr(A)^{\theta} \neq trA$

32. A is a skew-symmetric then is diagonal elements are all zero

33. The diagonal elements of a skew-symmetric matrix all are zero

34. Every diagonal element of Hermitian matrix must be real

35. The diagonal elements of skew- Hermitian matrix must be purely imaginary numbers or zero

36. For any non zero real k if A is symmetric (skew-symmetric) metric then kA is also symmetric (skew-symmetric)

37. If A is Hermitian Matrix then iA is skew Hermitian

38. If A is a skew-Hermitian matrix, iA is Hermitian

39. $(AB)^n = A^n B^n$

40. For real matrices A, B and C

- **a.** $A^T A = 0$ iff $A = 0$
- **b.** $AB = 0$ iff $A^T AB = 0$
- **c.** $AB = AC$ iff $A^T AB = A^T AC$

41. $(AB)^T = A^T B^T$ iff $AB = BA$

42. If A and B be diagonal matrices then AB is also diagonal and that $AB = BA$

43. Let A be square real matrix then

- **a.** $A + A^T$ is symmetric , even if A is not symmetric
- **b.** AB is not necessarily symmetric if A and B .... are
- **c.** $A^T BA$ is symmetric if B is symmetric but that the converse need not be true

44. Let A be a square real matrix

- **a.** $A - A^T$ is skew-symmetric
- **b.** A can be decomposed or the sum of a symmetric and a skew symmetric matrix

3.4. Properties on Determinants

1. The sum of the products of the elements of any row or column with the corresponding cofactors of the matrix is equal to the value of the determinant.

2. The sum of the products of the elements of any row and the cofactors of some other row is zero

3. Let $A = (a_{ij})$ be square matrix of order n, and let C_{ij} be the cofactor of a_{ij} then

- **a.** $|A| = \sum_{j=1}^{n} a_{ij} C_{ij} \, (i = 1, \ldots n)$
- **b.** $|A| = \sum_{j=1}^{n} a_{ij} C_{ij} \, (j = 1, \ldots n)$
- **c.** $\sum_{j=1}^{n} a_{ij} C_{kj} = 0$ if $i \neq k$
- **d.** $\sum_{i=1}^{n} a_{ij} C_{ik} = 0$ if $j \neq k$

4. The determinant of a triangular matrix is the product of if diagonal elements
5. For any two square matrices of the same order, $|AB| = |A||B|$
6. The value of the determinant of a diagonal matrix is equal to the product of the elements lying along its principal diagonal
7. The value of the determinant of a unit matrix is always equal to 1
8. The value of the determinant of triangular matrix (Upper or lower) is equal the product of the elements lying along its principal diagonal
9. The value of a determinant does not change when rows and columns are interchanged. $|A| = |A^T|$
10. If any two rows (or two columns) of a determinant are interchanged, the value of the determinant is multiplied by -1
11. If all the elements of one row (one column) of a determinant are multiplied by the some number k, the value of the row determinant is k times the value of determinant.
12. If A be an n-rowed square matrix, and k be any scalar, then $|kA| = k^n |A|$.
13. If two rows (or two columns) of a determinant are identical the value of the determinant is zero
14. The value of the determinant of a skew-symmetric odd order is always zero
15. If in a determinant each element in any row or column consists of the sum of two terms, then the determinant can be expressed or the sum of two determinant of the same order
16. If to the element of row (or columns) of a determinant added M times the corresponding elements of another row (or column) the value of the determinant thus obtained is equal to the value the original determinant
17. Let A be a Square matrix then
    - a. $|A^T| = |A|$
    - b. $|\overline{A}| = \overline{|A|}$
    - c. $|A^\theta| = \overline{|A|}$

3.5. Properties on Inverse of a Matrix

1. $(Adj\ A)A = A(Adj\ A) = |A| I_n$.
2. Every invertible matrix possess a unique inverse
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$
5. $(A^{-1})^\theta = (A^\theta)^{-1}$
6. $adj \cdot 0 = 0$
7. $adj \cdot I_n = I_n$
8. $adj\ A^T = (adj\ A)^T$

9. If A is a symmetric matrix, then $adj\,A$ is also symmetric

10. If A is symmetric then A^{-1} or also symmetric

11. The adjoint of a diagonal matrix of order 3 is a diagonal matrix

$$Let\,A\begin{bmatrix}\alpha & 0 & 0\\ 0 & m & 0\\ 0 & 0 & n\end{bmatrix} \qquad adj\,A=\begin{bmatrix}mn & 0 & 0\\ 0 & \alpha n & 0\\ 0 & 0 & \alpha m\end{bmatrix}$$

12. $\det\left(A^{-1}\right)=(\det A)^{-1}$

13. If B is a non-singular than the matrix A and $B^{-1}AB$ have the same $\det A$ & B of A

14. If the matrices A and B commute, then A^{-1} and B^{-1} also commute

15. If A,B,C be three matrices conformable for multiplication then $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$

16. If a matrix A is non-singular, than $AB=AC\Rightarrow B=C$, where B and C are square matrix of the same order of A

17. If the product of two non-zero square matrices is a zero matrix, then both of them must be singular.

18. If $|A|=0$, then $|adjA|=0$

19. $adj(AB)=adjB\cdot adjA$

20. If $A\cdot B$ be n-rowed orthogonal matrices AB and BA are also orthogonal matrices

21. If A,B be n-rowed unitary matrices, AB and BA are also unitary

3.6. Rank of a Matrix

1. The rank of a matrix A is r if it possess the following two properties
   - **a.** There is at least one square sub matrix of A of order r whose determinant is not equal to zero
   - **b.** And all the matrix A Counting any square sub matrix of order $r+1$, then the determinant of every square sub matrix of A of order $r+1$ should be zero.
2. The rank of matrix is the order of any highest order non-vanishing minor of the matrix.
3. The rank of every non-singular matrix of order n is n.
4. The rank of a square matrix A of order n less than n iff A is singular i.e. $|A|=0$
5. The rank of every non-zero matrix is ≥ 1
6. The rank of a matrix is $\leq r$, if all $(r+1)-rowed$ minors of the matrix vanish
7. The rank of a matrix is $\geq r$, if there exists $r-rowed$ minor of the matrix which is not equal to zero

8. The rank of a matrix in Echelon form is equal to the number of non-zero rows of the matrix.

9. The rank of the transpose of a matrix is same as that of original matrix.

10. The rank of matrix every element of which is unity is 1

11. A is a non-zero column and B a non-zero row matrix then rank$(AB)=1$

12. Rank of a matrix is $\geq$ the rank of every sub matrix

13. Rank of skew-symmetric matrix cannot be rank 1

14. Let A be an $m\times n$ matrix.

- **a.** $0\leq rank(A)\leq \min\{m,n\}$
- **b.** $rank(A)=0 \Leftrightarrow A=0$
- **c.** $rank(I_n)=n$
- **d.** $rank(kA)=k.rank(A) if\ k\neq 0$

15. Let A be an $m\times n$ matrix and let $Ax=0$ for some $x\neq 0 \Leftrightarrow rank(A)\leq n-1$

16. The rank of diagonal matrix equals the number of non-zero diagonal elements it possess but is not true for triangular matrix.

17. Every $m\times n$ matrix A of rank r can be written as $A=BC^T$, where $B(m\times r)$ and $C(m\times r)$ both have rank r

18. $rank(A+B)\leq rank(A)+rank(B)$

19. $rank(A-B)\geq |rank(A)-rank(B)|$

20. $rank(AB)\leq \min(rank(A),rank(B))$

21. $rank(AB)\neq rank(BA)$

22. Let A_1 be a sub matrix of A then $rank(A_1)\leq rank(A)$

23. $rank(A^TAB)=rank(AB)=rank(ABB^T)$ for any conformable matrices A and B.

24. $rank(A)=rank(A^T)=rank(AA^T)=rank(A^TA)$

25. Let A be an $m\times n$ matrix, and let $B(m\times n)$ and $C(m\times n)$ be non-singular

- **a.** $rank(BA)=rank(A)$
- **b.** $rank(AC)=rank(A)$
- **c.** $rank(BAC)=rank(A)$

26. For conformable matrices A,B,C

- **a.** $rank(BA)=rank(B^TBA)$
- **b.** $rank(AC)=rank(ACC^T)$
- **c.** $rank(BA)=rank(A)$
- **d.** $rank(AC)=rank(A)$

3.7. Properties on Trace

1. Let A and B be square matrices of the same order, and let $k \,\&\, m$ are positive scalars, then

   a. $tr(A+B) = tr(A) + tr(B)$

   b. $tr(kA) = ktr(A)$

   c. $tr(k.A + m.B) = k.tr(A) + m.tr(B)$

   d. $tr(A^T) = tr(A)$

   e. $tr(AA^T) = tr(A^T A) = \sum_{ij} a_{ij}^2$

2. For any real matrix A, $tr(A^T A) \geq 0$, *with* $tr(A^T A) = 0$ *iff* $A = 0$

3. Let A and B be $m \times n$ matrices, then $tr(A^T B) = tr(BA^T) = tr(AB^T) = tr(B^T A)$

4. $tr(ABC) = tr(CAB) = tr(BCA)$ but $tr(ABC) = tr(ACB)$ is not true

5. A real square matrix A is normal if $A^T A = AA^T$

   a. Every symmetric matrix is normal.

   b. Every orthogonal matrix is normal

   c. Let A be a normal 2×2 matrix then A is either symmetric or has the form $A = k\begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} (k \neq 0)$

6. A^p is symmetric when A is symmetric.

7. Consider an $n \times n$ triangular matrix A Then the all positive integral power of A are also triangular

8. For two real matrices A and B of the same order then Inner product is defined as $\langle A,B \rangle = \sum_i \sum_i a_{ij} b_{ij} = tr\, A^T B$

   a. $\langle A,B \rangle = \langle B,A \rangle$

   b. $\langle A,B+C \rangle = \langle A,B \rangle + \langle A,C \rangle$

   c. $\langle kA,B \rangle = k\langle A,B \rangle$

   d. $\langle A,A \rangle \geq 0$ with $\langle A,A \rangle = 0 \Leftrightarrow A = 0$

9. For a real matrix A the Norm of matrix is defined as $\|A\| : \langle A,A \rangle^{1/2} = \sqrt{trA^1 A} = \sqrt{\sum_i \sum_i a_{ij}^2}$

   a. $\|kA\| = |k| \cdot \|A\|$

   b. $\|A\| \geq 0.$ with $\|A\| = 0$ iff $A = 0$

   c. $\|A+B\| \leq \|A\| + \|B\|$ (*triangular inequlity*)

10. Let A and B are square matrix of same of order. $AB = I$ if $BA = I$

11. Let A be an orthogonal matrix of order n

- **a.** A is non-singular
- **b.** $A^T = A^{-1}$
- **c.** A^T is orthogonal
- **d.** AB is orthogonal when B is orthogonal

12. If A is orthogonal, $|A| = \pm 1$

13. If A is non-singular, $|A^{-1}| = |A|^{-1}$

14. Let $Z_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$, $Z_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ then

$$Z = Z_1 Z_2 = \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix}$$

15. $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}$

16. If A and D are square, not necessarily of the same order, $tr\begin{pmatrix} A & B \\ C & D \end{pmatrix} = trA + trD$

17. Let $Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ Where A and D are square matrices, not necessarily of the same order.

- **a.** If Z is symmetric then A and D are symmetric.
- **b.** If Z is diagonal then A and D are diagonal
- **c.** If Z is upper triangular then A and D are upper triangular
- **d.** All the above three statements are necessary but not sufficient

MATRICES

- Definitions ,Operations on Matrices, classification of Matrices,
- Determinant, Inverse of Square Matrix and its properties ,Trace of Matrix and its properties, Rank of Matrix and its properties,
- Solution of System of linear equations and its properties,

CHAPTER 4

EIGEN VALUES AND EIGEN FUNCTION

4.1 Definition

Let $T:V(F)\rightarrow V(F)$ be a linear transformation. then $\lambda\in F$ is called an eigen value if there exists non-zero $x\in V$ such that if $T(x)=\lambda x$, and the non-zero vector x is called eigen vector or eigen function.

Examples:

1. Let $V(\mathbb{R})$ be a vector space of all real valued function and $T:V(\mathbb{R})\rightarrow V(\mathbb{R})$ be L.T. such that $T(f)=f'$ then T has infinite eigen values
2. Let $V=\mathbb{R}[x]$ be a vector space of all polynomial functions with real coefficients and $T:V(\mathbb{R})\rightarrow V(\mathbb{R})$ be L.T. such that $T(f)=f'$ then 0is only eigen value of T, corresponding eigen functions are non-zero constant polynomial.
3. Let $V=\mathbb{R}[x]$ be a vector space of all polynomial functions with real coefficients and $T:V(\mathbb{R})\rightarrow V(\mathbb{R})$ be L.T. such that $T(f)=\int_0^x f(t)dt$ then T does not have any eigen value
4. Identity transformation on any non-trivial vector space $V(F)$ has one eigen value

Necessary and Sufficient Condition for Eigen Value

Let $V(F)$ be a vector space and $T:V(F)\rightarrow V(F)$ be linear operator then

1. $\lambda\in F$ is an Eigen value of T iff $T(x)=\lambda x$ for some $0\neq x\in V$
2. $\lambda\in F$ is an Eigen value of T iff $(T-\lambda I)x=0$ for some $0\neq x\in V$
3. $\lambda\in F$ is an Eigen value of T iff $x\in\ker(T-\lambda I)$ for some $0\neq x\in V$
4. $\lambda\in F$ is an Eigen value of T iff $(T-\lambda I)$ is not one-one
5. $\lambda\in F$ is an Eigen value of T iff $(T-\lambda I)$ is singular linear operator

Necessary and Sufficient Condition for Eigen Value for Finite Dimension Vector Space

Let $V(F)$ be a vector space over F of dimension n and $T:V(F)\rightarrow V(F)$ be linear operator. If A be the matrix representation of T corresponding to an ordered basis $\beta$of $V(F)$ then

1. $\lambda\in F$ is an Eigen value of A iff $\eta(A-\lambda I)>0$
2. $\lambda\in F$ is an Eigen value of A iff $\rho(A-\lambda I)<n$
3. $\lambda\in F$ is an Eigen value of A iff λ is root of the equation $det(A-\lambda I)=0$

Characteristic Polynomial: Let A be square matrix of order n then $C_A(x) = det(xI - A)$ is a polynomial of degree n called the characteristic polynomial of A and the Equation $C_A(x) = det(xI - A) = 0$ is called characteristic equation.

Proposition: 4.1 Let $V(F)$ be a vector space of dimension n over field F and $T : V(F) \to V(F)$ be linear operator if A be the matrix representation of T corresponding to an ordered basis β of $V(F)$ then $\lambda \in F$ is an Eigen value of A iff λ is root of the characteristic equation.

Cayley – Hamilton Theorem: every square matrix satisfies its characteristic equation i.e. if for a square matrix A of order n.

$|A - \lambda I| = (-1)^n [\lambda^n + a_1\lambda^{n-1} + \ldots . a_n]$ then

$A^n + a_1 A^{n-1} + \ldots\ldots + a_n I = 0$

Example: The characteristic equation of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

$= (2 - \lambda) \{(2-\lambda)^2 - 1\} + 1 \{-1 (2-\lambda) + 1\} + 1\{1-(2-\lambda)\}$

$= (2-\lambda) (3 - 4\lambda + \lambda^2) + (\lambda - 1) + (\lambda - 1)$

$= -\lambda^2 + 6\lambda^2 - 9\lambda + 4$

$\therefore$ The characteristic equation of the matrix A is

$\lambda^2 - 6\lambda^2 - 9\lambda - 4 = 0$

Proposition 4.1: Let $V(F)$ be a vector space of dimension n over field F and $T : V(F) \to V(F)$ be linear operator. If A be the matrix representation of T corresponding to an ordered basis β of $V(F)$ then $\lambda \in F$ is an Eigen value of A iff λ is root of the characteristic equation.

Eigen Space: If $T : V(F) \to V(F)$ be linear operator and $\lambda \in F$ is eigen value then $E_\lambda = \{x \in V \mid T(x) = \lambda x\}$ is subspace of $V(F)$ called eigen space corresponding to $\lambda \in F$. And every non-zero vector of E_λ is an eigen vector corresponding to $\lambda \in F$

Proposition on Eigen values and Eigen spaces:

Let $V(F)$ be a vector space over field F and $T : V(F) \to V(F)$ be linear operator.

Proposition 4.2: An eigen vector cannot correspond to two distinct eigen values, i.e, if I are two distinct eigen values then $E_\lambda \cap E_\mu = \{0\}$

Proposition 4.3: Eigen vectors corresponding to distinct eigen values are linearly independent .

Proposition 4.4: If $\lambda, \mu \in F$ are two distinct eigen values of T and $x, y \in V$ are eigen vectors corresponding to λ, μ respectively then for any $\alpha, \beta \in F - \{0\}$, $\alpha x + \beta y$ never Eigen vector corresponding to any eigen value.

In particular: If $\lambda, \mu \in F$ are two distinct eigen values and $x, y \in V$ are eigen vectors corresponding to λ, μ respectively then $x + y$ & $x - y$ are never eigen vector

Proposition 4.5: If linear operator $T : V(F) \to V(F)$ has two distinct eigen values $\lambda, \mu \in F$ then there must exist some non-zero vector $x \in V$ which are never eigen vector corresponding to any eigen value.

Proposition 4.6: Let $V(F)$ be a vector space over field F and $T : V(F) \to V(F)$ be linear operator. If every $0 \neq x \in V$ is an eigen vector then

1. T has Unique eigen value (say λ)
2. T is scalar multiple of I, i.e., $T = \lambda I$

Geometric multiplicity: Let $V(F)$ be a vector space over field F and $T : V(F) \to V(F)$ be linear operator. If λ is an eigen value of T then dimension of eigen space corresponding to λ is called geometric multiplicity of λ. I.e., the number of linearly independent vectors corresponding to λ is defined as geometric multiplicity of λ

OR Geometric multiplicity of an eigen $\lambda = dim(E_\lambda) = dim(ker(T - \lambda I))$ that is Geometric multiplicity of an eigen λ is Nullity of $(T - \lambda I)$

Proposition: 4.7: Let T_1 and T_2 linear operators on a finite dimensional vector space $V(F)$. If x is an eigen vector T_1, T_2 corresponding to eigen values λ_1, λ_2 respectively, then x is an eigen vector of $T_1 o T_2$ corresponding to eigen value $\lambda_1 \lambda_2$.

Proposition: 4.8: Let P and Q be two $n \times n$ matrices over a field F. If x is an eigen vector of P and Q corresponding to eigen values λ_1, λ_2 respectively. Then x is an eigen vector of PQ corresponding to eigen value $\lambda_1 \lambda_2$.

Proposition: 4.9: Let λ be an eigen value of a linear operator T over a vector space $V(F)$, then $f(\lambda)$ is an eigen value of $f(T)$ for any polynomial $f(x)$ over F.

Proposition: 4.10: Let $V(F)$ be a vector space of dimension n over field F and $T : V(F) \to V(F)$ be linear operator.if A be the matrix representation of T corresponding to an ordered basis $\beta$of $V(F)$ then

1. Sum of the all eigen values $= Trace(A)$
2. Product of the all eigen values $= det(A)$

Example: If 0, 1, -1 are eigenvalue of a matrix A the find $= det(I + A^{100})$

Proposition 4.11: Let $V(F)$ be a vector space of dimension n over field F and $T : V(F) \to V(F)$ be linear operator.if A be the matrix representation of T corresponding to an ordered basis $\beta$of $V(F)$ then If at least one eigen value of a matrix A is zero then the matrix A is singular and vice-versa.

Proposition 4.12: If A be the 3×3 matrix then

$$det(A-\lambda I)=0 \Rightarrow C_A(\lambda)=\lambda^3-Tr(A)\lambda^2+Tr(Adj(A))-det(A)=0$$

is characteristic polynomial of A

Algebraic Multiplicity: The order of λ as a root of characteristic polynomial is called algebraic multiplicity of λ

Proposition 4.13: If A be $n\times n$ matrix defined over the field F and $\lambda \in F$ an eigen value then

1. eigen value of A^{-1} is a λ^{-1} (provided A non-singular)
2. $\forall \alpha \in F$ eigen value of αA is $\alpha\lambda$
3. $\forall n \in \mathbb{N}$ an eigen value of A^n is λ^n
4. If $f(x)\in F[x]$ is a polynomial then $f(\lambda)\in F$ is an eigen value of $f(A)$
5. eigen value of $Adj(A)$ is $\dfrac{det(A)}{\lambda}$, provided $\lambda \neq 0$
6. If A and B are square matrices then AB and BA have same non-zero eigen values with algebraic multiplicity
7. If $A=\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ then the eigen values of $A=$ eigen values of B and eigen values of D

Proposition 4.14: S is an n-rowed real skew symmetric matrix and I is the unit matrix of order n. then

1. $I+S$ is a non-singular matrix
2. $A=(I-S)(I+S)^{-1}$ is an orthogonal matrix;
3. $A=(I+S)^{-1}(I-S)$
4. If $\lambda_1,\ldots,\lambda_n$ be the characteristic roots of S, then the characteristic roots of A are $(1-\lambda_1)(1+\lambda_1)^{-1}\ldots(1-\lambda_n)(1+\lambda_n)^{-1}$

Proposition 4.15: If A be an orthogonal matrix with the properly that -1 is not a characteristic root, then A is expressible as $(I-S)(I+S)^{-1}$ for some suitable skew symmetric matrix S.

Proposition 4.16: If A and B are non-singular matrices such that $AA^T=BB^T$ then there exists an orthogonal matrix P such that $A=BP$

Proposition 4.17: The geometric multiplicity of any Eigen value of an $n\times n$ matrix cannot exceed the algebraic multiplicity of the same eigen value.

Monic Polynomial: A polynomial in x in which the coefficient of the highest power of x is unity is called a monic polynomial.

Minimal Polynomial: If $f(x)$ is any polynomial S.t. $f(A)=0$ then we sayt hat $f(x)$ annihilates A The lowest degree mnic polynomial which annihilates A is asaid to be minimal polynomial of A.

Proposition 4.18: Let A be $n \times n$ matrix define over the field F and let $m(x)$ is the minimal polynomial then

1. Minimal polynomial of a matrix divides every polynomial, which annihilates the matrix.
2. Minimal polynomial of a matrix divides its characteristic polynomial.
3. The minimal polynomial of matrix is unique.
4. Every root of the characteristic polynomial of a matrix is also a root of minimal polynomial.
5. If the roots of the characteristic polynomial of a matrix are all distinct then the minimal polynomial and the characteristic polynomial are same.
6. λ is an eigen value iff $\lambda \in F$ is root of $m(x) = 0$

Derogatory and Non-derogatory Matrices: A matrix is non-derogatory if the degree of its minimal polynomial is equal to the degree of its characteristic polynomial. In other hand if the degree of minimal polynomial is less than the degree of characteristic polynomial then the matrix is called derogatory.

Gershgorin Theorem: The spectral radius of any matrix cannot exceed the maximum of absolute row sum or absolute column sum.

i.e. $A = \left[a_{ij}\right]$ $R_i = \sum_{j=1}^{n} |a_{ij}|$ & $C_j = \sum_{i=1}^{n} |a_{ij}|$ Then $|\lambda_i| \leq \max\left\{R_j\right\}$ or $|\lambda_i| \leq \max\left\{c_j\right\}$

Proposition 4.19 Let be the sum of the moduli of the elements along the k^{th} row excluding the diagonal element a_{kk}, then show that every eigen value of A lies inside or on the boundary of at least one of the circles $|\lambda - a_{kk}| = P_k$.

Proposition 4.20: Let $A = [a_{ij}]$ be $n \times n$ matrix then

1. If each row sum or column sum of A is α then α must be an Eigen value of A
2. If each row sum of A is α then α must be an Eigen value of A and $x = (1,1,\ldots,1)^T$ is corresponding Eigen vector
3. If each row sum of A is α then each row sum of A^k is α^k
4. If each row sum of A is α then sum of all entries of A^k is $n.\alpha^k$
5. If A is non-singular and each row sum of A is α then each row sum of A^{-1} is $\frac{1}{\alpha}$

4.2 Similarity of Matrices and Diagonalization

Let A and B are two matrices are square matrices A is called similar to B if there exist a non-singular matrix P such that $A = P^{-1}BP$

Orthogonally Similar Matrices: Let A and B are two matrices are square matrices A is called similar to B if there exist a orthogonal matrix P such that $A = P^T BP$

Diagonalizable Matrix: Let A any square matrix is called diagonalizable Matrix if it is similar to a diagonal matrix

Orthogonally Diagonalizable Matrix: Let A any square matrix is called orthogonally diagonalizable matrix if it is orthogonally similar to a diagonal matrix.

Properties on Similarity of Matrix:

1. $T:V_n(F)\to V_n(F)$ be linear operator and let β & β' are tow ordered basis if $A=[T:\beta,\beta]$ and $B=[T:\beta',\beta']$ Then A & B are similar matrices
2. If A is similar to a diagonal matrix, then A^T is similar to A.
3. Let A and B are square matrices and let A be non-singular than the matrices $A^{-1}B$ and $B^{-1}A$ have same Eigen values.
4. If A and B are non-singular matrices of order n, than the matrices AB and BA are similar.
5. The matrices AB and BA have same non-zero eigen value with multiplicity
6. A non-zero matrix is nilpotent iff all its Eigen values are equal to zero.
7. A & B are similar matrices then
    - (a) They have same determinate that is $\det(A)=\det(B)$
    - (b) They have same trace that is $Tr(A)=Tr(B)$
    - (c) They have same rank that is $\rho(A)=\rho(B)$
    - (d) They have same nullity that is $\eta(A)=\eta(B)$
    - (e) They have same characteristic polynomial that is $C_A(x)=C_B(x)$
    - (f) They have same minimal polynomial that is $m_A(x)=m_B(x)$
    - (g) They have same Eigen values

Diagonalization of Matrices: If $T:V_n(F)\to V_n(F)$ be linear operator then T is called diagonalizable if there exist a basis such that βsuch that the corresponding matrix representation is diagonal matrix or

A matrix A is said to be diagonalizable iff it is similar to a diagonal matrix i.e. $\exists$ a non-singular matrix P s.t. $P^{-1}AP$ is a diagonal matrix.

Example: Out matrix $A=\begin{bmatrix}-6 & 6\\ -12 & 11\end{bmatrix}$ is diagonalizable, the eigenvectors

$\begin{bmatrix}3\\4\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$ are linearly independent, and so form a basis for $\mathbb{R}$. Indeed,

we see that $A=\begin{bmatrix}-6 & 6\\ -12 & 11\end{bmatrix}\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}=\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$

so that $P^{-1}AP$ is diagonal, where P is the matrix whose columns are the eigenvectors of A.

Furthermore, one can find two projection matrices whose column spaces are the eigenspaces, namely

$$P_1=\begin{bmatrix}9 & -6\\ 12 & -8\end{bmatrix},\quad P_2=\begin{bmatrix}-8 & 6\\ -12 & 9\end{bmatrix}$$

Theorem 4.10: The linear map α on V is diagonalizable if and only if its minimal polynomial is the product of distinct linear factors, that is, its roots all have multiplicity 1.

Proof: Suppose first that α is diagonalizable, with eiganvalues $\lambda_1,....,\lambda_r$. Then there is a basis such that α is represented by a diagonal matrix D whose diagonal entries are the eigenvalues. Now for any polynomial f, $f(\alpha)$ is represented by $f(D)$, a diagonal matrix whose diagonal entries are $f(\lambda_i)$ for $i=1,....,r$. Choose

$$f(x)=(x-\lambda_1)....(x-\lambda_r).$$

Then all the diagonal entries of $f(D)$ are zero; so $f(D)=0$. We claim that f is the minimal polynomial of α; clearly it has no repeated roots, so we will be done. We known that each λ_i is a roots of $m_\alpha(x)$, so that $f(x)$ divides $m_\alpha(x)$; and we also know that $f(\alpha)=0$, so that the degree of f cannot be smaller than that of m_α. So the claim follows.

Conversely, we have to show that if m_α is a product of distinct linear factors then α is diagonalizable. This is a little argument with polynomial. Let $f(x)=\prod(x-\lambda_i)$ be the minimal polynomial of α, with the roots λ_i all distinct. Let $h_i(x)=f(x)/(x-\lambda_i)$. Then the polynomials $h_1,....,h_r$ have no common factor except 1; for the only possible factors are $(x-\lambda_i)$, but this fails to divide h_i. Now the Euclidean algorithm shows that we can write the h.c.f. as a linear combination:

$$1=\sum_{i=1}^{r}h_i(x)k_i(x).$$

Let $U_i=\text{Im}(h_i(\alpha)$. The vectors in U_i are eigenvectors of α with eigenvalue λ_i; for if $u\in U_i$, say $u=h_i(\alpha)v$, then

$$(\alpha-\lambda_i I)u_i=(\alpha-\lambda_i I)h_i(\alpha)(v)=f(\alpha)v=0$$

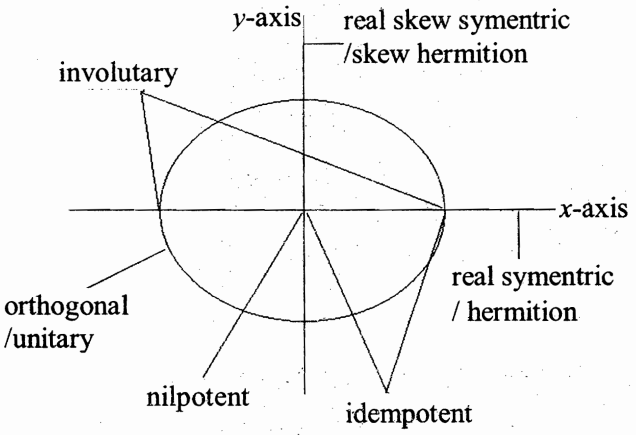
So that $\alpha(v)=\lambda_i(v)$. Moreover every vector can be written as a sum of vectors from the subspaces U_i. For, given $v\in V$, we have

$$v=Iv=\sum_{i=1}^{r}h_i(\alpha)(k_i(\alpha)v),$$

with $h_i(\alpha)(k_i(\alpha)v)\in\text{Im}(h_i(\alpha)$. The fact that the expression is unique follows from the lemma, since the eigenvectors are linearly independent.

Properties Diagonalization of Matrices: Let $T : V_n(F) \to V_n(F)$ be linear operator on vector space $V_n(F)$ then T

1. T is diagonalizable over field F iff there exist a basis such that β consisting Eigen vectors only
2. T is diagonalizable over field F iff algebraic multiplicity of each eigen value is same as geometric multiplicity each eigen value provided all the root of characteristic polynomial are in field.
3. T is diagonalizable over field F iff it has n linearly independent Eigen vectors where n is order of the matrix.
4. If the sum of dimensions of the Eigen spaces of a matrix equals the order of that matrix then the matrix can be diagonalized over field F and conversely. i.e., if $\sum \dim(W_{\lambda_i}) = n$, where W_{λ_i} are Eigen spaces corresponding to Eigen values λ_i I.e. sum of G.M. of all Eigen values is n.
5. If all the Eigen values of a matrix are distinct then the matrix is diagonalizable over field F but converse may or may not be.
6. A matrix is diagonalizableover field F iff the minimal polynomial contains only non-repeatedlinear factors.
7. A nilpotent matrix is diagonalizableover field F iff it is a null matrix.
8. Spectrum of Eigen values



9. Eigen values of real symmetric matrix or Hermitian matrices are real
10. Eigen values of real Skew symmetric matrix or Skew Hermitian matrices are purely imagery or zero
11. Eigen values of Unitary Matrix or real Orthogonal matrix are unit modulo one
12. Eigen values Idempotent Matrix or orthogonal Projection matrix either zero or one
13. Eigen values **Involutory matrices** or orthogonal reflection matrix: either 1 or -1

14. Eigen values Nilpotent Matrix are zero

15. Eigen values of permutation matrix are roots of unity

4.3 Important Linear Operator / Matrices

1. If all the entries of a matrix are equal say '$\alpha \neq 0$' then matrix can be over field F diagonalized i.e. if $A = [a_{ij}]$; $a_{ij} = \alpha \; \forall \; i, j$

(a) Then the eigen values of $2 \times 2p$ are 0 & $n\alpha$

(b) $\rho(A) = 1$ & $\eta(A) = n - 1$

(c) $G.M \;\; of \; 0 = n - 1 = A.M \;\; of \; 0$ and $G.M \;\; of \;\; n\alpha = 1 = A.M \;\; of \;\; n\alpha$

(d) A is diagonalizable

(e) $C_A(x) = x^{n-1}(x - Tr(A))$

(f) $m_A(x) = x(x - Tr(A))$

2. $A = [a_{ij}]; \sum_{i=1}^{n} a_{ij} = 1 \; \forall \; j = 1, 2, ..n \in \{0, 1\}$ such that $\sum_{i=1}^{n} a_{ij} = 1$, $\forall \; j = 1, 2, ..n$

and $\sum_{j=1}^{n} a_{ij} = 1$, $\forall i = 1, 2, ..n$

That is A permutation matrix

(a) Each permutation matrices of order n correspondence to $\sigma \in S_n$

(b) Number of permutation matrices of order n are $n!$

(c) $\det(A) = sig(\sigma)$ where $\sigma \in S_n$ is permutation corresponding to matrix A

(d) $Tr(A) =$ Number of symbols which are fixed (self-imagedin σ). where $\sigma \in S_n$ is permutation corresponding to matrix A

(e) A is orthogonal /unitary matrix

(f) A is diagonalizable over complex field

(g) If $\sigma = c_1.c_2.....c_k$ product of disjoint cycles such that $l(c_i) = r_i$ then $A^k = I$, where k is the order of permutation

(h) $\sigma = c_1.c_2.....c_k$ product of disjoint cycles such that $l(c_i) = r_i$ then characteristic of A is $C(x) = \prod_{r_i} (x - 1)^{r_i}$

(i) $\sigma = c_1.c_2.....c_k$ product of disjoint cycles such that $l(c_i) = r_i$ then eigen values of A are r_i -th roots of unity

(j) The Algebraic multiplicity of each eigen value is same as geometric multiplicity

(k) The algebraic multiplicity of an eigen value k^{th} -root of unity is number of disjoint cycles whose length is multiple of k

(l) A is diagonalizable over real field iff σ is either of order one or of order 2

Example: $A=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdot^{\cdot^{\cdot}} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ be $n\times n$ matrix then A is diagonalizable over real field

3. $A=a^T b$ where $a=(a_1,a_2,\ldots a_n)\in\mathbb{R}^n$ and $b=(b_1,b_2,\ldots b_n)\in\mathbb{R}^n$

Then

(a) $\rho(A)=1$ & $\eta(A)=n-1$

(b) Eigen values of A are 0 and $Tr(A)$

(c) $G.M\ \ of\ 0=n-1=A.M\ \ of\ 0$ and $G.M\ \ of\ \ n\alpha=1=A.M\ \ of\ \ n\alpha$

(d) A is diagonalizable iff $Tr(A)\neq 0$

(e) $C_A(x)=x^{n-1}(x-Tr(A))$

(f) $m_A(x)=x(x-Tr(A))$ provided $Tr(A)\neq 0$

(g) A is nilpotent iff $Tr(A)=0$

4. $A^2=A$ be $n\times n$ idempotent matrix Then

(a) Then minimal polynomial of A can be $x\ or\ x-1\ or\ x(x-1)$ in each of the case minimal polynomial contains non-repeated linear factors

(b) Eigen value of A are either $0\ or\ 1$

(c) $0\le Tr(A)\le n$, and $Tr(A)\in\mathbb{N}\cup\{0\}$

(d) If $Tr(A)>n\ or\ Tr(A)<0\ or\ Tr(A)\notin\mathbb{N}\cup\{0\}$ then A never idempotent

(e) A is diagonalizable over every field implies $A.M.$ of each eigen value is same as $G.M.$

(f) $Tr(A)=\rho(A)$ and $\eta(A)=n-Tr(A)$

(g) $C(A)=x^{n-Tr(A)}(x-1)^{Tr(A)}$

5. If $T:V(F)\to V(F)$, linear operator where $dim(V)=n$ such that $T^2=T$ then

(a) $n=\eta(I-T)+\eta(T)$

(b) $n=\rho(I-T)+\rho(T)$

(c) $\rho(I-T)=\eta(T)$

(d) $\eta(I-T)=\rho(T)$

(e) $Image(T-I)=Ker(T)$

(f) $Image(T)=Ker(T-I)$

(g) $V=Ker(T-I)+Ker(T)$

(h) $V=Ker(T-I)+Image(T-I)$

(i) $V=Image(T)+Ker(T)$

(j) $V=Image(T)+Image(T-I)$

6. $A^2=I$ be $n\times n$ involuntary matrix Then

(a) Then minimal polynomial of A can be $x+1$ *or* $x-1$ *or* $(x+1)(x-1)$ in each of the case minimal polynomial contains non-repeated linear factors

(b) Eigen value of A are either -1 *or* 1

(c) $-n\le Tr(A)\le n$, and $Tr(A)\in\mathbb{Z}$

(d) If $Tr(A)>n$ *or* $Tr(A)<-n$ *or* $Tr(A)\notin\mathbb{Z}$ then A never idempotent

(e) A is diagonalizable over every field implies $A.M.$ of each eigen value is same as $G.M.$

(f) $Tr(A)=\eta(A-I)-\eta(A+I)$

(g) $Tr(A)=\rho(A+I)-\rho(A-I)$

(h) $n=\rho(A-I)+\eta(A-I)$

(i) $n=\rho(A+I)+\eta(A+I)$

(j) $\rho(A+I)=\eta(A-I)$

(k) $\rho(A-I)=\eta(A+I)$

(l) $n=\eta(A-I)+\eta(A+I)$

(m) $\dfrac{n+Tr(A)}{2}=\eta(A-I)$

(n) $C(A)=(x-1)^{\frac{n+Tr(A)}{2}}(x+1)^{\frac{n-Tr(A)}{2}}$

7. If $T:V(F)\to V(F)$, linear operator where $dim(V)=n$ such that $T^2=I$ then

$Image(T-I)=Ker(T+I)$

$Image(T+I)=Ker(T-I)$

$V=Ker(T-I)+Ker(T+I)$

$V=Ker(T+I)+Image(T+I)$

$V=Ker(T-I)+Image(T-I)$

$V=Image(T-I)+Image(T-I)$

4.4 Canonical Forms

Invariance: Let $T:V\to V$ be linear. A subspace W of V is said to be invariant under T or T-invariant if T maps W into itself, i.e., if v∈W implies $T(v)W$. In this case T restricted to W defines a linear operator on W; that is , T induces a linear operator $\hat{T}:W\to W$ defined by $\hat{T}$ $(w) = T(w)$ for every $w\in W$.

Example: Nonzero eigenvectors of linear operator $T:\ V \to V$ may be characterized as generators of T-invariant 1-dimensional subspaces. For suppose $T(v) = \lambda v,\ v \neq 0$. Then W = {kv, k ∈ K}, W = {kv, k ∈ K} the 1-dimen-sional subspace generated by v, is invariant under T.

Proposition 4.16: Let $T:\ V \to V$ be linear, and let f(t) be any polynomial. Then the kernel of $f(T)$ is invariant under T.

Proposition 4.17: Suppose W is an invariant subspace of $T:\ V \to V$. Then T has a block matrix representation $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A is a matrix representation of the restriction of T to W.

Invariant Direct-sum Decompositions: A vector space V is termed the direct sum of its subspaces $W_1, \ldots, W_r$, written $V = W_1 \oplus W_2 \oplus \ldots\ldots\ldots \oplus W_r$ If every vector v∈V can be written uniquely in the form $v = w_1 + w_2 + \ldots\ldots + w_r$ with $w_i \in W_i$

Proposition 4.18: Suppose $T: V \to V$ is linear and V is the direct sum of T-invariant subspace $W_1, \ldots\ldots, W_r$. If A_i is a matrix representation of the restriction of T to W_i, then T can be represented by the block diagonal matrix.

$$M = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_r \end{pmatrix}$$

The block diagonal matrix M with diagonal entries $A_1, \ldots\ldots, A_r$ is sometimes called the direct sum of the matrices $A_1, \ldots\ldots, A_r$ and denoted by $M = A_1 \oplus \ldots\ldots \oplus A_r$.

Proposition 4.19: If T be a linear operator V such that $T^k(v) = 0$ for some $v \in V$ and $T^{k-1}(v) \neq 0$ and v is called generalized Eigen vector of order k.

1. The set $S = \{v,\ T(v),\ T^2(v),\ -----\ T^{k-1}(v)\}$ *is L.I.*
2. The subspace W, generated by S is T-invariant
3. The restriction $\hat{T}: W \to W$ is nilpotent of index k.
4. The matrix of $\hat{T} = \begin{bmatrix} 0 & 0 & 0 --- & 0 & 0 \\ 1 & 0 & 0 --- & 0 & 0 \\ 0 & 1 & ------ & & | \\ | & : & ------ & & | \\ 0 & 0 & 0 --- & 1 & 0 \end{bmatrix}$

4.5 Jordan Canonical Form

Block Matrix: A matrix of form $A = \begin{bmatrix} C & D \\ P & Q \end{bmatrix}$, where C, D, P, Q are also matrices then matrix A is called block matrix.

Block Diagonal Matrix: A matrix of the form $A = \begin{bmatrix} A_{11} & O & O \\ O & A_{22} & O \\ O & O & A_{33} \end{bmatrix}$ is called block diagonal matrix where A_{ii} are square matrices may not be of same order and are called block diagonal.

Proposition 4.20:

1. If $Ch_{A_{ii}}(x)$ is characteristic polynomial of A_{ii} then $Ch_A(x)$ the characteristic polynomial of A is given as $Ch_A(x) = \prod Ch_{A_{ii}}(x)$ also $\sum O(A_{ii}) = O(A)$.
2. If $M_{A_{ii}}(x)$ is minimal polynomial of A_{ii} then $M_A(x)$ the minimal polynomial of A is given as $M_A(x) = l.c.m.\ of\ M_{A_{ii}}(x)$

Bi-diagonal Matrix: A matrix $A = [a_{ij}]_{n\times n}$ is bi-diagonal if it is defined as follows

$$A = \begin{cases} a_{ii} = k & \forall\ i = 1\,to\,n \\ a_{ii+1} = 1 & \forall\ i = 1\,to\,n-1 \end{cases}$$

Example: $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ is a bi-diagonal matrix.

Jordan Block: A bi-diagonal matrix is said to be Jordan Block if all its diagonal entries are equal e.g. $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is a Jordon block of order 4.

Jordan Canonocal Form: A Block diagonal matrix is called in its J.C. form if their block diagonals are Jordan block having Eigen values on main diagonal.

Example: $A = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{vmatrix}$ is J.C. form of matrix with Eigen values 2, 3, 5.

Proposition 4.21:

1. Every matrix can be converted to J.C. form.
2. Every matrix is similar to its J.C. form.
3. Every matrix has its unique J.C. form.
4. Two matrices are similar iff they have same J.C. form.

Method to write J.C. Form: Let $A=[a_{ij}]_{n\times n}$ be the matrix s.t.

$$Ch_A(x)=\prod_{i=1}^{k}(x-\alpha_i)^{n_i}$$

$\sum_{i=1}^{k} n_i = n$ and $M_A(x)=\prod_{i=1}^{k}(x-\alpha_i)^{m_i}$: $\sum_{i=1}^{k} m_i \le n$ then following conditions are to be applied

1. In the J.C. form for each α_i there is at least one block of order m_i and other blocks corresponding to same α_i have order less than or equal to m_i.
2. Sum of order of all blocks corresponding to α_i is equal to n_i.
3. The number of Jordan block blocks corresponding to α_i is equal to G.M. of α_i.
4. If the transformation is known then the J.C. form is uniquely determined.

Example:

Let $Ch_A(x)=(x-2)^3(x-3)^3(x-1)^2$ and

$M_A(x)=(x-2)^2(x-3)^2(x-1)$ then its J.C. form is

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Proposition: 4.22

1. If $Ch_A(x)$ and $M_A(x)$ or both are given for an unknown matrix A then J.C. form may not be obtained uniquely but have always finite number of possibilities of J.C. form of such matrix.
2. If $Ch_A(x)=\prod_{i=1}^{k}(x-\alpha_i)^{n_i}$ then number of possible J.C. form is $\prod P(n_i)$ where $P(n_i)$ denotes the number of partitions of n_i.

   Example: If $Ch_A(x)=(x-2)^5(x-3)^5$ then number of possible J.C. form is $P(5).P(5)=7.7=49$

Put Your Own Notes

3. If $Ch_A(x)=\prod_{i=1}^{k}(x-\alpha_i)^{n_i}$ and $M_A(x)=\prod_{i=1}^{k}(x-\alpha_i)^{m_i}$ and let $n_i-m_i=t_i$ then number of possible J.C. forms $\prod S(t_i)$ where $S(t_i)$ denotes the number of partitions of t_i in which no part is greater then m_i.

   Example: $Ch_A(x)=(x-2)^7$ and $M_A(x)=(x-2)^2$ then $n-m=7-2=5=t$ Now $S(5)=3$ thus number of possible J.C. form is 3.

4. It is not always possible to find G.M. of a matrix out of given characteristic polynomial and minimal polynomial.

5. If A is a nilpotent matrix of index k then its J.C. form must have at least one block of order k and all other blocks will be of order less than or equal to k.

6. The number of blocks in case of nilpotent matrix gives the nullity of matrix.

7. Number of distinct $n\times n$ matrices up to similarity with given characteristic polynomials are number of possible J.C. forms.

CHAPTER 5

INNER PRODUCT AND ITS PROPERTIES

5.1 Inner Product

If u and v are vectors in R^n, then we regard u and v as $n \times 1$ matrices. The transpose u^T is a $1 \times n$ matrix, and the matrix of product $u^T v$ is a 1×1 matrix, which we write as a single real number (a scalar without brackets. The number $u^T v$ is called the inner product of u and v, and often it is written as $u.v$.

$$u = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_n \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{bmatrix}$$

Then the inner product of u and v is written as $< u\ ,\ v >$ and

$$< u,\ v > \ = [u_1 \quad u_2 \quad \ldots \quad u_n] \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

Theorem: Let u, v and w be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

(a) $u.\,v \;=\; v.\,u$

(b) $(u+v).\,w = u.\,w \;+\; v.\,w$

(c) $(cu).v \;=\; c(u.\,v) \;=\; u.(cv)$

(d) $u\ .\ u^3 0$, and $u.\,u \;=\; 0$ if and only if $u = \;0$

Properties (b) and (c) can be combined several times to produce the following useful rule:

$$(c_1 u_1 + \;\ldots\; + c_p u_p)\ .w \;=\; c_1(u_1.\,w) \;+\; \ldots \;+\; c_p(u_p.\,w)$$

Some Important Inner Products

i. Consider the dot product in R^n:

$$u\ .\ v \;=\; a_1 b_1 + a_2 b_2 + \;\ldots\; + a_n b_n$$

Where $u = \{a_1, a_2, ..., a_n\}$ and $v = \{b_1, b_2, ..., b_n\}$ This as an inner product on $\mathbb{R}^n$, and $\mathbb{R}^n$ with this inner product is usually referred to as Euclidean n-space. Although there are many different ways to define an inner product on $\mathbb{R}^n$. We shall assume this inner product on $\mathbb{R}^n$ unless otherwise stated or implied.

ii. Consider the dot product on $\mathbb{C}^n$:

$$u . v = u.v = z_1\overline{w}_1 + z_2\overline{w}_2 + ... + z_n\overline{w}_n$$

Where $u = \{z_1, z_2, ..., z_n\}$ and $v = \{w_1, w_2, ..., w_n\}$ As in the real case, this is an inner product on $\mathbb{C}^n$ and we shall assume this inner product on $\mathbb{C}^n$ unless otherwise stated or implied.

iii. Let V denote the vector space of $m \times n$ matrices over $\mathbb{R}$. The following is an inner product in V :

(A, B) = tr (B^tA),Wheretr stands for trace, the sum of the diagonal elements.

Analogously, if U denotes the vector space of $m \times n$ matrices over $\mathbb{C}$, then the following is an inner product in U : $(A, B) = tr(B * A)$

As usual, $B *$ denotes the conjugate transpose of the matrix B.

iv. Let V be the vector space of real continuous functions on the interval $a \le t \le b$. Then the following is an inner product on V :

$$(f, g) = \int_a^b f(t) g(t) dt$$

Analogously, if U denotes the vector space of complex continuous functions on this (real) interval $a \le t \le b$, then the following is an inner product on U :

$$(f, g) = \int_a^b f(t) \overline{g(t)} dt$$

5.2 Inner Product Space

The vector space $V(F)$ with an inner product is called inner product space.

A real inner product space is called an Euclidean space and a complex inner product space is called unitary space.

Example: Let V be the vector space of infinite sequences of real numbers $(a_1, a_2, \ldots)$ satisfying

$$\sum_{i=1}^{\infty} a_i^2 = a_1^2 + a_2^2 + ... < \infty$$

i.e. the sum converges. Addition and scalar multiplication are defined component wise:

$$(a_1, a_2, ...) + (b_1, b_2, ...) = (a_1 + b_1, a_2 + b_2, ...)$$

$$k (a_1, a_2, ...) = (ka_1, ka_2, ...)$$

An inner product is defined in V by

$$\langle (a_1, a_2, ...), (b_1, b_2, ...) \rangle = a_1b_1 + a_2b_2 + ...$$

The above sum converges absolutely for any pair of points in V. Hence the inner product is well defined. This inner product space is called l_2-space (Hilbert space)

5.3 Some Important Definitions

The Length of A Vector: If v is in $\mathbb{R}^n$, with entries $v_1, \ldots, v_n$, then the square root of (v, v) is defined because $v.v$ is nonnegative.

So the length (norm) of v is the nonnegative scalar $\| v \|$ defined by

$$\| v \| = \|v\| = \sqrt{v.v} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}\,, \quad \text{and} \quad \|v\|^2 = v.v$$

For any scalar c, the length of cv is $|c|$ times the length of v. That is $\|cv\| = |c|\|v\|$

Distance Between Two Vectors: For u and v in $\mathbb{R}^n$, the distance between u and v, written as dist(u, v), is the length of the vector $u - v$. That is dist$(u, v) = \| u - v \|$

Orthogonal Vectors: Two vectors u and v in $\mathbb{R}^n$ are orthogonal (to each other) if $u \,.\, v = 0$. Observe that the zero vector is orthogonal to every vector in $\mathbb{R}^n$ because $0^T v = 0$ for all v.

Theorem: The Pythagorean Theorem

Two vectors u and v are orthogonal if and only if $\| u + v \|^2 = \| u \|^2 + \| v \|^2$.

Angles Between Two Vectors: If u and v are non zero vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$, then there is a nice connection between their inner product and the angle θ between the two line segments from the origin to the points identified with u and v. The formula is

$$u \,.\, v = \| u \| \, \| v \| \, \cos q$$

To verify this formula for vectors in $\mathbb{R}^2$, consider the triangle shown in figure, with sides of length $\| u \|$, $\| v \|$, and $\| u - v \|$. By the law of cosines,

$$\| u - v \|^2 = \| u \|^2 + \| v \|^2 - 2\| u \| \, \| v \| \, \cos q$$

Which can be rearranged to produce

$$\|u\| \, \|v\| \cos\theta = \frac{1}{2}[\|u\|^2 + \|v\|^2 - \|u - v\|^2]$$

$$= \frac{1}{2}[u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2]$$

$$= u_1 v_1 + u_2 v_2$$

$$= u.\, v$$

Orthonormal Sets: A set $\{u_1, u_2, ..., u_n\}$ of vectors in V is said to be orthogonal if its distinct elements are orthogonal, i.e. if $\langle u_i, u_j \rangle = 0$ for $i \neq j$. In particular, the set $\{u_1, u_2, ..., u_n\}$ is said to be orthonormal if it is orthogonal and if each u_i has length 1, that is, if

$$\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 0 & for \quad i \neq j \\ 1 & for \quad i = j \end{cases}$$

An orthonormal set can always be obtained from an orthogonal set of nonzero vectors by normalizing each vector.

1. Let V be the vector space of real continuous functions on the interval $a \le t \le \pi$ with inner product defined by $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) g(t) dt$. The following is a classical example of an orthogonal subset of V:

   $\{1, \cos t, \cos 2t, ..., \sin t, \sin 2t, ...\}$

2. Consider the usual basis of Euclidean 3-space $\mathbb{R}^3$:

   $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$

   It is clear that

   $\langle e_1, e_2 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1$ and $\langle e_i, e_j \rangle = 0$ $for\, i \neq j$

That is $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathbb{R}^3$. More generally, the usual basis of $\mathbb{R}^n$ or of $\mathbb{C}^n$ is orthonormal for every n.

Orthogonal Complements: To provide practice using inner products, we introduce a concept here.. If a vector z is orthogonal to every vector in a subspace W of R^n, then z is said to be orthogonal to W. The set of all vectors z that are orthogonal to W is called the orthogonal complement of W and is denoted by $W^\perp$ (and read as "W perpendicular" or simply " W perp").

1. A vector x is in $W^\perp$ if and only if x is orthogonal to every vector in a set that span W.

2. If S is any set of vectors in an inner product space $V(F)$, then $W^\perp$ is a subspace of V.

Theorem: Let W be a subspace of V. Then V is the direct sum of W and $W^\perp$, i.e.

$$V = W \oplus W^\perp$$

Now if w is a subspace of V, then $V = W \oplus W^\perp$ by the above theorem; hence there is a unique projection $E_w : V \to V$ with image W and kernel $W^\perp$. That is, if $v \in V$ and where $w \in W$, $w' \in W^\perp$, then E_w is defined by $E_w(v) = w$. This mapping E_w is called orthogonal projection of V onto W.

Example: Let W be the z axis in R^3, i.e. $W = \{(0,\ 0,\ c) : c \in R\}$

Then $W^{\perp}$ is the xy plane, i.e. $W^{\perp} = \{(a,\ b,\ 0) :\ a,\ b \in R\}$

As noted previously, $R^3 = W \oplus W^{\perp}$.

The orthogonal projection E of R^3 onto W is given by $E(x,\ y,\ z) = (0,\ 0,\ z)$

Orthogonal Complement of an Orthogonal Complement: If S is a subset of an inner product space $V(F)$ then $S^{\perp}$ is a subspace of V Therefore,

$$(S^{\perp})^{\perp} = \{\alpha \in V \ | \alpha.\beta = 0, \forall b \in S^{\perp}\}$$

We write $(S^{\perp})^{\perp}$ *as* $S^{\perp\perp}$. It is a subspace of V and $S \subseteq S^{\perp\perp}$, S is called orthogonal complement of orthogonal complement $S^{\perp}$.

Results:

1. If V(F) is an inner product space and $S,\ S_1,\ S_2$ are subsets of V, then
   - **(i)** $\{0\}^{\perp} = V$
   - **(ii)** $V^{\perp} = \{0\}$
   - **(iii)** S and $S^{\perp}$ are disjoint, i.ee., $S \cap S^{\perp} = \{0\}$
   - **(iv)** $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp} = \{0\}$
   - **(v)** $S \subseteq S^{\perp\perp}$
   - **(vi)** $S^{\perp} = S^{\perp\perp\perp}$ where $S^{\perp\perp\perp} = (S^{\perp\perp})^{\perp}$.
2. If W is a subspace of finite dimensional inner product space V, then
   - **(i)** $V = W \oplus W^{\perp}$
   - **(ii)** $W^{\perp\perp} = W$
3. If W is a subspace of a finite dimensionalinner product space $V(F)$, then $\dim V = \dim W + \dim W^{\perp}$

Orthogonal Projections: If W is a subspace of a finite dimensional inner product space V, then $V = W \oplus W^{\perp}$ (proved above). Therefore, every vector $\alpha \in V$ can be uniquely expressed as $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in W$ and $\alpha_2 \in W^{\perp}$. The vectors α_1 and α_2 are called the orthogonal projections of α on the subspaces W and $W^{\perp}$.

Result: If W_1 and W_2 are subspaces of a finite dimensional inner product space V, then

(i) $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$

(ii) $(W_1 \cap W_2)^{\perp} = (W_1 + W_2)^{\perp}$

Inner Product Space Isomorphism and Unitary Operations: Let U and V be two inner product vector spaces over the same field F and let T be a linear transformation from U to V. We say

(i) T preserves inner products if $T(\alpha).T(\beta) = \alpha.\beta \ \forall \alpha, \beta \in U$.

(ii) T preserves norms if $\| T(\alpha) \| = \|\alpha\| \ \forall \alpha \in U$.

(iii) T preserves distances (i.e., T is isometry) if

$\| T(\alpha) - T(\beta) \| = \|\alpha - \beta\|, \forall \alpha, \beta U$ Where $d(T(\alpha), T(\beta)) = \| T(\alpha) - T(\beta) \|$

(iv) T is an inner product space isomorphism if it is invertible (i.e, one-one and onto) and preservers inner products.

Note:

(i) An isomorphism from U onto V is a linear transformation from U onto V which preserves inner products.

(ii) If T is an isomorphism from U onto V then T^{-1} is an isomorphism from V to U.

(iii) If T is an isomorphism from U onto V then we say that U and V are isomorphic and we write U = V

(iv) One can show that the relation of isomorphism of inner product space is an equivalence relation.

(v) Isomorphism of an inner product space onto itself is called a unitary operator on that space.

Theorem: Let U, V be two inner product vector spaces over the same field F. Let T be a linear transformation from U to V. Then, following three conditions are equivalent:

(i) T preserves inner products

(ii) T preserves norms

(iii) T is an isometry

Theorem: Let U and V be two finite dimensional inner product spaces over the same field and having the same dimension. If T is a linear transformation from U onto V, the following are equivalent.

(i) T preserves inner products

(ii) T is an inner product space isomorphism

(iii) T carries every orthonormal basis for U onto an orthonormal basis for V

(iv) T carries some orthonormal basis for U onto an orthonormal basis for V.

(v) If U and V are two finite dimensional inner product spaces over the same field, then U and V are isomorphic iff dim U = dim V.

Unitary Operator

A linear operator T on an inner product space $V(F)$ is called a unitary operator if T is an inner product space isomorphism of V onto itself. We can also say that T is a unitary operator if it is invertible and preserves inner products.

Results:

1. A linear operator T on a finite dimensional inner product space V is unitary iff T carries an orthonormal basis of V onto an orthonormal basis of V.
2. If V is a unitary vector space of dimension n with an inner product, then V is isomorphic to $V_n(C)$ with standard inner product.

Examples:

1. Define orthogonal complement of an inner product space. Prove that if S is any set of vectors of an inner product space $V(F)$, then the orthogonal complement of S is a subspace of V.

   Solution: See definition theorem 2 in the text book.

2. Define orthogonal complement of an orthogonal complement. If $S,\ S_1,\ S_2$ are subsets of an inner product space V then prove

   (i) $S \cap S^{\perp} = \{0\}$

   (ii) $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$

   (iii) $S \subseteq S^{\perp\perp}$

   Solution:

   (i) To Prove : $S \cap S^{\perp} = \{\bar{0}\}$.

   Let $x \in S \cap S^{\perp} \Rightarrow x \in S$ and $x \in S^{\perp}$

   $\Rightarrow \quad x \perp x \quad \Rightarrow x\,.\,x = 0 \Rightarrow x = \bar{0}$

   (ii) To Prove: $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \in S_1^{\perp}$

   Let $x \in S_2^{\perp} \Rightarrow x\,.\,y = 0 \quad \forall y \in S_2$

   Now, $S_1 \in S_2$

   $\therefore \quad y \in S_1 \quad \Rightarrow \quad y \in S_2$

   $\therefore \quad x \in S_2^{\perp} \quad \Rightarrow \quad (x\,.\,y) = 0 \quad \forall y \in S_1$

   $\Rightarrow \quad x \in S_1^{\perp}$

   $\therefore \quad x \in S_2^{\perp} \quad \Rightarrow x \in S_1^{\perp}$

   Hence , $S_2^{\perp} \subseteq S_1^{\perp}$

   (iii) To Prove: $S \subseteq S^{\perp\perp}$

   By definition, $S^{\perp} = \{x \ : \ x \perp S,\ i.e., (x,\ y) = 0\ \forall y \in S\}$

   $\therefore$ the relation $(x,\ y) = 0\, or\ (y,\ x) = 0$

   $\Rightarrow$ y is orthogonal to all vectors x which belong to $S^{\perp}$ and as such $y \in (S^{\perp})^{\perp} = S^{\perp\perp}$

   Hence, $y \in S \quad \Rightarrow \quad y \in S^{\perp\perp}$

   $\therefore \quad S \subseteq S^{\perp\perp}$.

If W is a subspace of a finite dimensional inner product space V, then prove that

(i) $V = W \oplus W^{\perp}$

(ii) $W^{\perp\perp} = W$.

Solution:

(i) Let us first prove that $V = W + W^{\perp}$

Since W is a subspace of a finite dimensional vector space V,

$\therefore$ W is also finite – dimensional

Let dim. $V = n$ and dim $W = m$ Now, every finite-dimensional vector space possesses an orthonormal basis.

Let $B = \{x_1, x_2, \ldots\ldots, x_m\}$ be an orthonormal basis for W. Let y be any vector in V.

Let $z = y - \sum_{i=1}^{m} \langle y, x_i \rangle x_i$ Then z is orthogonal to each of the vectors $x_1, x_2, \ldots\ldots\ldots, x_m$

[$\because$ If $S = \{x_1, x_2, \ldots\ldots, x_m\}$ is an orthonormal set in V and if $y \in V$ then $z = y - \sum_{i=1}^{m} \langle y, x_i \rangle x_i$ isorthogonal to each of $x_1, x_2, \ldots, x_m$]

$\because$ z is orthogonal to the subspace spanned by these vectors

$\because$ z is orthogonal to W

$\because$ $z \in W^{\perp}$ (by definition of orthogonal complement)

Also, the vector $\sum_{i=1}^{m} \langle y, x_i \rangle x_i$ being a linear combination o basic vector x_i, is in W.

$\because$ from (1), $y = \left(\sum_{i=1}^{m} \langle y, x_i \rangle x_i \right) + z$

Where $\sum_{i=1}^{m} \langle y, x_i \rangle x_i$ and $z \in W^{\perp}$

$\because$ $y \in W + W^{\perp}$

$\because$ $V = W + W^{\perp}$ **(i)**

Now, we shall show that $W \cap W^{\perp} = \{\overline{0}\}$

Since $x \in W^{\perp}$, x is orthogonal t every vector in W.

$\Rightarrow$ x is orthogonal to x because $x \in W$

$\because$ $<x, x> = 0 \Rightarrow x = \overline{0}$

Hence, $W \cap W^{\perp} = \{\overline{0}\}$ **(ii)**

From (i) and (ii), if follows that $V = W \oplus W^{\perp}$.

(ii) Since $V = W \oplus W^{\perp}$ and $W^{\perp}$ is a subspace of V

$\because$ taking $W^{\perp}$ is place of W, we get

$V = W^{\perp} \oplus (W^{\perp})^{\perp}$

i.e., $V = W^{\perp} \oplus W^{\perp\perp}$ **(1)**

Since V is the direct sum of W and $W^{\perp}$ and V is finite dimensional, therefore

$\dim V = \dim W + \dim W^{\perp}$ **(2)**

Similarly from (1), $\dim V = \dim W^{\perp} + \dim W^{\perp\perp}$ **(3)**

$\because$ from (2) and (3)

$\dim W = \dim W^{\perp\perp}$ **(4)**

Now, it remains to show that $W \subseteq W^{\perp\perp}$

Let $\alpha \in W \Rightarrow \alpha.\beta = 0 \ \forall \beta \in W^{\perp}$

$\therefore$ by definition of $(W^{\perp})^{\perp}$, we have $\alpha \in (W^{\perp})^{\perp}$

$\therefore \alpha \in W \Rightarrow \alpha \in (W^{\perp})^{\perp}$

$\therefore W \subseteq W^{\perp\perp}$

Since $W \subseteq W^{\perp\perp}$ therefore W is a subspace of $W^{\perp\perp}$

$\therefore$ from (4) and (5), $W = W^{\perp\perp}$ **(5)**

3. If W_1 and W_2 are subspaces of a finite dimensional inner product space V, then

(i) $(W_1 + W_2)^{\perp\perp} = W_1^{\perp} \cap W_2^{\perp}$

(ii) $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$

Solution:

(i) Since $W_1 \in W_1 + W_2$

$\therefore \quad (W_1 + W_2)^{\perp} \subseteq W_1^{\perp}$ **(1)**

$(\therefore S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp})$

Again since $W_2 \subseteq W_1 + W_2$

$\therefore (W_1 + W_2)^{\perp} \subseteq W_2^{\perp}$ **(2)**

$\therefore$ from (1) and (2),

$(W_1 + W_2)^{\perp} \subseteq W_1^{\perp} \cap W_2^{\perp}$ **(3)**

Now, we shall show that $W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1 + W_2)^{\perp}$

Let $a \in W_1^{\perp} \cap W_2^{\perp}$

$\Rightarrow a \in W_1^{\perp}$ and $a \in W_2^{\perp}$

$\Rightarrow \alpha.\beta = 0$ and $\alpha.\gamma = 0 \quad \forall \beta \in W_1$ and $\gamma \in W_2$

$\Rightarrow (\alpha.\beta + \gamma) = 0$ where $\beta + \gamma \in W_1 + W_2$

$\Rightarrow \alpha \in (W_1 + W_2)^{\perp}$

$\therefore W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1 + W_2)^{\perp}$ **(4)**

$\therefore$ from (3) and (4),

$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$ **(5)**

(ii) Since $W_1^{\perp}$ and $W_2^{\perp}$ are subspaces of V.

$\therefore$ replacing W_1 and W_2 by $W_1^{\perp}$ and $W_2^{\perp}$ in (5), we get

$$(W_1^{\perp} + W_2^{\perp})^{\perp} = W_1^{\perp\perp} \cap W_2^{\perp\perp}$$

$$= W_1 \cap W_2 \qquad (W^{\perp\perp} = W)$$

$$\therefore \quad (W_1^{\perp} + W_2^{\perp})^{\perp} = (W_1 \cap W_2)^{\perp}$$

$$\Rightarrow \quad W_1^{\perp} + W_2^{\perp} = (W_1 \cap W_2)^{\perp}.$$

4. Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T

   Then$(Row\ A)^{\perp} = Nul\ A$ *and* $(Col\ A)^{\perp} = Nul\ A^T$

Proof: The row – column rule for computing Ax shows that if x is in Nul A, then x is orthogonal to each row of A (with the rows treated as vectors in R^n). Since the rows of A span the row space, x is orthogonal to Row A. Conversely, if x is orthogonal to Row A, then x is certainly orthogonal to each row of A, and hence $Ax = 0$. This proves the first statement of the theorem. Since this statement is true for any matrix, it is true for A^T. That is, the orthogonal complement of the row space of A^T is the null space of A^T. This proves the second statement, because Row A^T = Col A.

5.4 Quick Review on Inner Product Spaces

1. **Pythagoras Theorem:** If $x \perp y$ then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.
2. **Cauchy Schwarz Inequality:** $|<x,y>| \le \|x\|\|y\| \quad \forall\ x, y \in V$ equality holds iff x and y are L.D.
3. **Triangular Inequality:** $\|x+y\| \le \|x\| + \|y\|$ if equality holds then x and y areL.D. but may not conversely.
4. $\|\|x\| - \|y\|\| \le \|x\| + \|y\| \quad \forall\ x, y \in V$
5. Every Orthonormal set in I.P.S. is L.I.

6. If $F=\mathbb{R}$ then $<x,y>=\frac{1}{2}\|x+y\|^2-\frac{1}{2}\|x\|^2-\frac{1}{2}\|y\|^2$

$=\frac{1}{4}\|x+y\|^2-\frac{1}{4}\|x-y\|^2$

7. If $F=\mathbb{C}$ then

$$<x,y>=\frac{1}{4}\|x+y\|^2-\frac{1}{4}\|x-y\|^2+\frac{i}{4}\|x+iy\|^2-\frac{i}{4}\|x-iy\|^2$$

8. **Projection Theorem:** Let W be a subspace of a finite dimensional V.S. over F then $V=W\oplus W^{\perp}$.

9. If W is subspace of finite dimensional V.S. then $W=W^{\perp\perp}$.

10. If W is $T-inveriant$ subspace then $W^{\perp}$ is also $T-inveriant$.

11. Let V be any I.P.S. S_1,S_2,S_3 are subspaces of V following holds

(i) $S_1\subseteq S_2\Rightarrow S_2^{\perp}\subseteq S_1^{\perp}$

(ii) $S^{\perp}=(L(s))^{\perp}$

(iii) $L(S)\subseteq S^{\perp}$

(iv) $L(S)=S^{\perp\perp}$ if E is finite dimensional

12. Let W_1 and W_2 be subspaces of finite dimensional I.P.S. then

(i) $(W_1+W_2)^{\perp}=W_1^{\perp}\cap W_2^{\perp}$

(ii) $(W_1\cap W_2)^{\perp}=W_1^{\perp}+W_2^{\perp}$

13. Let $S=\{e_1,e_2,...,e_n\}$ be any finite orthonormal set in I.P.S. E then following are equivalent :-

(i) S is complete

(ii) If $<x,e_i>=0$ for $i=1,2,3,...,n$ then $x=0$

(iii) $L(S)=E$

(iv) If $x\in E$, $x=\sum_{i=1}^{n}<x,e_i>e_i$

(v) If $x,y\in E$ then $<x,y>=\sum_{i=1}^{n}<x,e_i><e_i,y>$

(vi) For $x\in E$, $\sum_{i=1}^{n}|<x,e_i>|^2=\|x\|^2$

CHAPTER 6

BILINEAR FORM AND QUADRATIC FORMS

6.1 Bilinear Form

In those notes we introduce the notion of a dual space. Dual spaces are useful in that they allow as to phrase many important concepts in linear algebra without the need to introduce additional structure. In particular, we will see that we can formulate many notions involving inner products in a way that does not require the use of an inner product.

Let V be a vector space over a field F. Recall the following definition:

Definition 1. A linear functional on V is a linear map $V \to F$. In other words, a linear functional on V is an element of $\mathcal{L}(V,F)$.

Being examples of linear maps, we can add linear functional and multiply them by scalars. Also, there is a unique linear functional on V, called the zero functional, which sends everything in V to zero. All this gives the set of linear functional the structure of a vector space.

Definition 2. The dual space of V, denoted by V^*, is the space of all linear functional on V; i.e., $V^* := \mathcal{L}(V,\mathbb{F})$.

Proposition 1. Suppose that V is finite-dimensional and let $(v_1,\cdots,v_n)$ be a basis of V. For each $i=1,\cdots,n,$ defined a linear function $f_i : V \to F$ by setting $f_i(v_j)=\begin{cases}1 & if\ i=j\\ 0 & if\ i\neq j\end{cases}$

and then extending f_i linearly to all V. Then $(f_1,\cdots,f_n)$ is a basis of V^*, called the dual basis of $(v_1,\cdots,v_n)$. Hence, V^* is finite-dimensional and dim $V^* = \dim V$.

Proof: First we check that $(f_1,\cdots,f_n)$ is linearly independent. Suppose that $a_1,\cdots,a_n \in \mathbb{F}$ are scalars so that $a_1f_1+\cdots+a_nf_n=0$

Note that the 0 on the right denotes the zero functional; i.e. the functional which sends everything in V to $0\in\mathbb{F}$. The above equality above is an equality of maps, which should hold or any v we evaluate either side on. In particular, evaluating both sides on v_i, we have

$$(a_1f_1+\cdots+a_nf_n)(v_i)=a_1f_1(v_i)+\cdots+a_nf_n(v_i)=a_i$$

on the left (by the definition of the f_i) and 0 on the right. Thus we see that $a_i=0$ for each i, so $(f_1,\cdots,f_n)$ is linearly independent.

Now we show that $(f_1,\cdots,f_n)$ spans V^*. Let $f\in V^*$. For each i, let b_i denote the scalar $f(v_i)$/ We claim that $f=b_1f_1+\cdots+b_nf_n$.

Again, this means that both sides should give the same result when evaluating on any $v \in V$. By linearlity, it suffices to check that is true on the basis $(v_1, \ldots, v_n)$. Indeed, for each i we have

$$(b_1 f_1 + \cdots + b_n f_n)(v_i) = b_1 f_1(v_i) + \cdots + b_n f_n(v_i) = b_i = f(v_i),$$

again by the definition of the f_i and the b_i. Thus, f and $b_1 f_1 + \cdots + b_n f_n$ agree on the basis, so we conclude that they are equal as elements of V^*. Hence $(f_1, \cdots, f_n)$ spans V^* and therefore forms a basis of V^*.

Note in particular the following consequence of the above construction of the dual basis. If $v \in V$ is an element such that $f(v) = 0$ for all $f \in V^*$, then $v = 0$. To see this, let $(v_1, \cdots, v_n)$ be a basis of V and let $(f_1, \cdots, f_n)$ be the dual basis. Write v as $v = a_1 v_1 + \cdots + a_n v_n$.

By assumption, we have that $f_i(v) = 0$ for all i. But by the definition of f_i, $f_i(v) = a_i$. Thus $a_i = 0$ for all i and so $v = 0$ as claimed.

Let V^{**} denote $(V^*)^*$ — i.e. the dual space of the dual space of V, often called the double dual of V. If V is finite-dimensional, then we know that V and V^* are isomorphic since they have the same dimension. However, in general writing down an actual isomorphism between V and V^* require choosing a basis of V and constructing the dual basis of V^* — the required isomorphism the sends the ith basis vector of V to the corresponding dual basis vector of V^*. Similarly, since dim V^* also equals him V^{**}, we known that V and V^{**} are isomorphic. In this case however, there is an isomorphism between V and V^{**} which can be written down without the choice of a basis — such an isomorphism is said to be natural.

Theorem 1. Let V be an inner product space. Define a map $T : V \to V^*$ by i.e. Tv is the linear functional on V whose value on $w \in V$ is (v, w). Then T is an isomorphism.

This is done of the main conceptual uses of inner products — they allow us to identity a vector space with its dual in a natural way, where again natural means "without the choice of a basis".

Now we look at maps betweem dual spaces.

Definition 3. Let $T : V \to W$ be linear. The dual map (or transpose) of T is the map $T^* : W^* \to V^*$ defined by

$T^* g = gT$ for all $g \in W^*$.

In other words, T^* sends a linear functional g on W to the composition gT, which is a linear functional on V.

Proposition 3. Suppose that V is finite-dimensional and let $(v_1, \ldots, v_n)$ be a basis of V. Let $(f_1, \ldots, f_n)$ be the corresponding dual basis of V^*. For $T \in \mathcal{L}(V)$, the matrix of $T^* \in \mathcal{L}(V^*)$ with respect to $(f_1, \ldots, f_n)$ is the transpose of the matrix of T with respect to $(v_1, \ldots, v_n)$; i.e.

$$M(T^*) = M(T)^t$$

This gives the "true" meaning of taking the transpose of the matrix: if a matrix represents an operator on some finite-dimensional space, then its transpose represents the dual operator. Note that under the identification of V with V^* given by an inner product, the dual map corresponds exactly to the adjoint of T with respect to that inner product; indeed, this is why the dual map and adjoint use the same notation.

Finally, we give one last useful construction. Note again that under the identification of a vector space with its dual given by an inner product, the following corresponds to taking the orthogonal complement of a subspace. As above, this is why the following construction uses the same notation as the orthogonal complement.

Definition 4. Let U be a subspace of V. The annihilator of U in V^*, denoted by $U^{\perp}$, is the set of linear functional on V which vanish on U; i.e.

$$U^{\perp}=\{f\in V^* \mid f(u)=0 \text{ for all } u\in U\}$$

Note the $U^{\perp}$ is a subspace of V^*. Indeed, the zero functional sends everything in U to zero and so is in $U^{\perp}$, if $f(u)=0$ and $g(u)=0$, then $(f+g)(u)=f(u)+g(u)=0+0=0$ so $U^{\perp}$ is closed under addition, and finally if also $a\in\mathbb{F}$, then $(af)(u)=af(u)=a0=0$ so $U^{\perp}$ is closed under scalar multiplication.

Using annihilators, we can given a nice description of the dual space to a quotient space.

Proposition 4. There exists a natural isomorphism between $U^{\perp}$ and $(V/U)^*$. Hence we can identify linear functional on V/U with elements of $U^{\perp}$.

Proof. Let $f\in U^{\perp}$. Again, this means that f is a linear functional on V which vanishes on the subspace U. Define a linear functional Tf on V/U by $(Tf)(v+U)=f(v)$;

in other words, Tf sends the coset $v+U$ to the scalar $f(v)$. First we need to know that this definition of Tf is well-defined. Suppose that $u+U=v'+U$. We must check that evaluating Tf on either one gives the same result. Since $v+U=v'+U, v-v'\in U$. Thus since f vanishes on U, we have

$$0=f(v-v')=f(v)=f(v')$$

so $f(v)=f(v')$, showing that Tf is well- defined.

This then defines a map $T:U^{\perp}\to(V/U)^*$. From the definition of addition and scalar multiplication of liner functional, it follows that T is linear. We claim that T is invertible.

To show that T is injective, suppose that $f\in$ null T. Then Tf is the zero functional on V/U

So, $0=(Tf)(v+U)=f(v)$ for all $v\in V$.

This f is the zero functional on V(i.e. the zero element of $U^{\perp}$) so null $T=\{0\}$ and T is injective.

Finally, to so that T is surjective (note that we are not assuming V is finite – dimensional),

Let $g\in(V/U)^*$.Define an element $f\in V^*$ by

$f(v)=g(v+U)$ for all $v\in V$

We claim that f is actually in $U^{\perp}$. Indeed, if $v\in U$, then $g(v+U)=g(U)=0$ since U is the zero element of V/U and g is liner. Thus $f(v)=0$, so $f\in U^{\perp}$. By definition of T, it follows that $Tf=g$ so that T is surjective and thus invertible.

A Bilinear Form

Definition: Let F be a field and V be a vector space over F. A bilinear from on V is a function $B:V\times V\to F$ that is linear in each variable when the other one is fixed. That is,

$B(v+v',w)=B(v,w)+B(v',w)$, $B(cv,w)=cB(v,w)$

for all $v,v'w\in V$ and $c\in F$ and

$B(v,w+w')=B(v,w)+B(v,w')$, $B(v,cw)=cB(v,w)$,

for all $v,w,w'\in V$ and $c\in F$.

We call B symmetric when

$B(v,w)=B(w,v)$ for all $v,w\in V$

and skew –symmetric when

$B(v,w)=-B(w,v)$ for all $v,w\in V$

We call B alternating when

$B(v,v)=0$ for all $v\in V$

A bilinear space is a vector space equipped with a specific choice of bilinear form. We call a bilinear space symmetric, skew-symmetric, or alternating when the chosen bilinear form has that corresponding property.

A common synonym for skew-symmetric is anti symmetric.

Example: The dot product v. w on $\boldsymbol{R}^n$ is a symmetric bilinear form.

Example: For a fixed matrix $A\in M_n$ (R), the function $f(v,w)=v.Aw$ on $\boldsymbol{R}^n$ is a bilinear form, but not necessarily symmetric like the dot product. All later examples are essentially generalizations of this construction.

Example: For any field F, viewed as a 1-dimensional vector space over itself, multiplication $m:F\times F\to F$ is a symmetric bilinear form and not alternating. It is skew-symmetric when F has characteristic 2.

Example: A skew-symmetric and alternating bilinear form on $\boldsymbol{R}^2$ is

$$B((x,y),(x',y')) := xy' - x'y = \det\begin{pmatrix} x & x' \\ y & y' \end{pmatrix}.$$

For example, $B((2,1),(3,4)) = 5$ and $B((2,1),(2,1)) = 0$. Viewing $\boldsymbol{R}^2$ as C by

$(x,y) \leftrightarrow x+iy, B(z,w) = \operatorname{Im}(\bar{z}w) = -\operatorname{Im}(z\bar{w})$ for complex numbers z and w.

Among the three types of bilinear forms we have defined (symmetric, skew-symmetric, alternating), the first and third types are more basic than the second. In fact, we now show that a skew-symmetric bilinear form is just another name for a symmetric or an alternating bilinear form, depending on whether or not the characteristic of the field is 2.

Theorem: In all characteristics, an alternating bilinear form is skew-symmetric. In characteristic not 2, a bilinear form is skew-symmetric if and only if it is alternating. In characteristic 2, a bilinear form is skew-symmetric if and only if it is symmetric.

Proof: When B is alternating and $v, w \in V$, expanding right side of the equation $0 = B(v+w, v+w)$

$$0 = B(v,v) + B(v,w) + B(w,w) = B(v,w) + B(w,v)$$

So $B(v,w) = -B(w,v)$. Therefore alternating bilinear forms are skew-symmetric in all characteristics (even in characteristic 2). Outside of characteristic 2, a skew-symmetric bilinear form is alternating since

$$B(v,v) = -B(v,v) \Rightarrow 2B(v,v) = 0 \Rightarrow B(v,v) = 0.$$

That skew-symmetric and symmetric bilinear forms coincide in characteristic 2 is immediate sine $1 = -1$ in characteristic 2.

Theorem: In characteristic not 2, every bilinear form B is uniquely expressible as a sum $B_1 + B_2$, where B_1 is symmetric and B_2 is alternating (equivalently, skew-symmetric).In characteristic 2.the alternating bilinear forms are a subset of the symmetric bilinear forms.

Proof. Now we work in characteristic not 2. For a bilinear form B, suppose we can write $B = B_1 + B_2$ with symmetric B_1 and alternating (so skew-symmetric) B_2. Then for vectors v and w,

$$(1.1)\quad B(w,v) = B_1(w,v) + B_2(w,v)$$

$$(1.2)\quad = B_1(v,w) + B_2(v,w)$$

Adding and subtracting 1.1 and 1.2, we get formulas for B_1 and B_2 in terms of B:

$$(1.3)\quad B_1(v,w) = \frac{B(v,w)+B(w,v)}{2}, B_2(v,w) = \frac{B(v,w)-B(w,v)}{2}.$$

Turning this reasoning around, the bilinear forms B_1 and B_2 defined by 1.3are symmetric and alternating

respectively, so we have established the existence and uniqueness of B_1 and B_2.

Theorem: In characteristic not 2, a symmetric bilinear form $B(v,w)$ is completely determined by its

Values $B(v,v)$ on the diagonal.

Proof : For any v and w.

$$\frac{1}{2}\left(B(v+w,v+w)-B(v,v)-B(w,w)\right)=\frac{1}{2}\left(B(v,w)+B(v,w)\right)=B(v,w)$$

Note we used symmetric of B in the last equation.

The fact that, for symmetric B, we can recover the 2-variable function $B(v,w)$ from the 1- variable

Function $B(v,v)$ outside of characteristic 2 is called polarization. For instance, it show us that a

symmetric bilinear form B is identically 0 if and only if $B(v,v)=0$ for all v(not just $B(e_{i,}e_i)=0$ on a basis;

Let's look at some more examples of bilinear forms

From now on, all vector spaces are understood to be finite dimensional.

A linear transformation $L:V\rightarrow W$ between two finite dimensional vector spaces over F can be written as a matrix once we pick (ordered) bases for V and W. When V = W and we use the same basis for the inputs and outputs of L then changing the basis leads to a new matrix representation that is conjugate to the old matrix. In particular, the trace, determinant, and (more generally) characteristic polynomial of a linear operator $L:V\rightarrow W$ are well-defined, independent of the choice of basis. In this section we will see how bilinear forms and related constructions can be described using matrices.

We start with a concrete example. In addition to the dot product on $\boldsymbol{R}^n$, additional bilinear forms on $\boldsymbol{R}^n$

Are obtained by throwing a matrix into one side of the dot product: for an $n\times n$ real matrix M, the formula

$B(w,v)=v\cdot Mw$ is a bilinear form on $\boldsymbol{R}^n$. It turns out this kind of construction describes all bilinear forms

on $\boldsymbol{R}^n$. It turns out this kind of construction describes all bilinear forms on any finite- dimensional vertor

space, once we fix a basis.

Let V have dimension $n\geq 1$ with basis $\{e_1,\cdots,e_n\}$. Pick v and w in V and express them in this basis:

$\{e_1....,e_n\}$. Pick v and w in V and express them in this basis: $v=\sum_{i=1}^{n}xiei$ and $w=\sum_{j=1}^{n}yiei$. For any

Bilinear form B on V, its bilinearity gives

$$B(v,w)=B\left(\sum_{i=1}^{n}x_i,ej,\sum_{j=1}^{n}y_je_j\right)$$

$$=\sum_{i=1}^{n} x_i B\left(e_i, \sum_{j=1}^{n} y_j e_j\right)$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j B\left(e_i, e_j\right)$$

Set $M := \left(B(ei, ej)\right)$, which is an $n \times n$ matrix. By a calculation the reader can carry out,

(21.) $B(v,w)=[v].M[w]$

For all v and w in V, where on the right is the usual dot product on F^n and

$$[v]=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, [w]=\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

are the coordinate vectors of v and w for w for our choice of basis $\{e_1,\cdots,e_n\}$. The "coordinate" isomorphism $[\,]:V \to F^n$ will be understood to refer to a fixed choice of basis throughout given discussion.

We call the matrix $M=\left(B\left(e_i,e_j\right)\right)$ appearing in 2.1 the matrix associated to B in the basis $\{e_1,\cdots,e_n\}$.

Example: The matrix associated to the dot product on F^n in the standard basis of F^n is the identity matrix.

Example: In Example

$$xy'-x'y=\begin{pmatrix} x \\ y \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

It is easy to read off the matrix from formula on the left: there are no xx' or yy' terms, so the diagonal entries of the matrix are 0. Since xy' has coefficient 1, the (1,2) entry of the matrix is 1. The term $x'y = yx'$ corresponds to the (2,1) entery (because it involves the second of x and y and the first of x' and y', in the order) which must be the coefficient -1.

Theorem: Let V be a vector space over F of dimension $n \geq 1$. For a fixed choice of basis $\{e_1,...,e_n\}$ of V, which given an isomorphism $v \leftrightarrow [v]$ form V to F^n by coordination, each bilinear form on V has the expression (2.1) for a unique $n \times n$ matrix M over F.

Proof: We have shown every bilinear form looks like (2.1) once we choose a basis. It remains to verify uniqueness. Suppose $B(v,w)=[v].N=[w]$ for some matrix N. Then $B\left(e_i,e_j\right)=[e_i].N\left[e_j\right]$, which is the (i,j) entry of N, so $N=\left(B\left(e_i,e_j\right)\right)$.

Note the zero vector space has 1 bilinear form but no matrix. We will not be pedantic about including the zero vector space in our discussion.

Example: Let V= $\boldsymbol{R}^n$. Pick nonnegative integers p and q such that $p+q=n$. For $v=(x_1,\cdots,x_n)$ and $v'(x'_1,\cdots,x'_n)$ in ʔ^n , set

$$\langle v,v'\rangle_{p,q} := x_1x'_1+\cdots+x_px'_{p+1}-\cdots-x_nx'_n$$

$$= v\cdot\begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix}v'.$$

This symmetric bilinear form is like the dot product, except the coefficients involve p plus signs and $n-p=q$ minus signs. The dot product on $\boldsymbol{R}^n$ is the special case $(p,q)=(n,0)$.

Example: Is the special case $(p,q)=(1,1)$.

The space $\boldsymbol{R}^n$ with the bilinear form $\langle\cdot,\cdot\rangle_{p,q}$ is denoted $\boldsymbol{R}^{p,q}$. We call $\boldsymbol{R}^{p,q}$ a psecudo-Edclidean space when p and q are both positive. The example $\boldsymbol{R}^{1,3}$ or $\boldsymbol{R}^{3,1}$ is called Minkowski space and arises in relativity theory. A pseudo-Euclidean space is the same vector space as $\boldsymbol{R}^n$, but its geometric structure (e.g., the notion of perpendicularity) is different. The label Euclidean space actually not just another name for $\boldsymbol{R}^n$ as a vector space, but it is the name for $\boldsymbol{R}^n$ equipped with a specific bilinear form: the dot product.

Theorem: Let (V,B) be a bilinear space and let B have associated matrix M in some basis. Then

(1) B is symmetric if and only if $M^T=M$

(2) B is skew-symmetric if and only if $M^T=-M$

(3) B is alternating if and only if $M^T=-M$ and the diagonal entries of M are zero

6.2 Quadratic Forms

Congruence Matrices and Quadratic Form

Congruence Matrices: A matrix A is said to be congruent to B if there exist $P(\ |P|\neq 0\ )$ s.t.

$$A\ =\ PBP^T\ i.e.\ A\equiv B$$

Quadratic Form: An expression of the form $\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j$,

Where a_{ij}'s are elements of a field F , is called a quadratic form in then n variables $x_1,\ x_2,\ ....,\ x_n$ over a field F.

* Each expression of quadratic form can be written as a symmetric matrix

Example:

If $f(x,\ y,\ z)\ =\ ax^2+\ by^2+2hxy\ +\ 2gxz\ +2fyz\ +cz^2$ then it can be written in matrix form i.e.

$$f(x,\ y,\ z)\ =\ [x,\ y,\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Reduction of Quadratic Forms

Even if we cannot find a canonical form for quadratic forms, we can simplify them very greatly.

Theorem 5.6:

Let q be a quadratic form in n variable $x_1,\ldots\ldots,x_n$, over a field $\mathbb{K}$ whose characteristic is not 2. Then by a suitable linear substitution to new variables $y_1,\ldots.,y_n$, we can obtain.

$q = c_1y_1^2 + c_2y_2^2 + \ldots. + c_ny_n^2$ for some $c_1,\ldots\ldots,c_n \in \mathbb{K}$.

Example 5.1:

Consider the quadratic form $q(x, y, z) = x^2 + 2xy + 4xz + y^2 + 4z^2$.

we have $(x+y+2z)^2 = x^2 + 2xy + 4xz + y^2 + 4z^2 + 4yz$,

and so $q = (x+y+2z)^2 - 4yz$

$= (x+y+2z)^2 - (y+z)^2 + (y-z)^2 = u^2 + v^2 - w^2,$

where $u = x+y+2z,\quad v = y-z,\quad w = y+z$. Otherwise said, the matrix representing the quadratic form, namely $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ is congruent to the matrix $A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ Can you find an invertible matrix P such that $P^T AP = A'$?

6.3 Definite, Semi-Definite and Indefinite Quadratic Forms, Value Class of a Form

Let $\phi = x'Ax$ be a real quadratic form. The form ϕ is said to be

(i) Positive definite, if $\phi > 0$ for all real values of the variables other than $x_1 = \ldots.. = x_n = 0$, e.g., $x_1^2 + x_2^2 + x_3^2$ is a positive definite form in 3 variables;

(ii) Positive semi-definite, if $\phi \geq 0$ for all real values of the variables other than $x_1 = \ldots = x_n = 0$; e.g., $x_1^2 + x_2^2$ is a positive semi-definite form in three variables;

(iii) Negative –definite, if $\phi < 0$ for all real values of the variables other than $x_1 = \ldots = x_n = 0$; e.g., $-x_1^2 - x_2^2 - x_3^2$ is a negative definite form in 3 variables;

(iv) Negative semi-definite, if $\phi \leq 0$ for all real values of the variables other than $x_1 = \ldots x_n = 0$; e.g., $-x_1^2 - x_3^2$ is a negative semi-definite form in 2 variables;

(v) Indefinite, if ϕ takes positive as well as negative values for real values of the variables; e.g., $x_1^2 - x_2^2$ is an indefinite form.

A form which is positive definite or negative definite is called definite and a form which is positive semi-definite or negative semi-definite is called semi-definite.

The above five classes of real quadratic forms are mutually exclusive and are called value-classes of real quadratic forms. The set of all real quadratic forms can be partitioned into these classes i.e. every real quadratic form must belong to one and only one value class.

Value-Class of the Quadratic Form

$\phi = y_1^2 + ... + y_p^2 - y_{n+1}^2 - ... - y_r^2$, in n variables.

We shall show that the value class of ϕ depends on p and r. It will, however , be more convenient to set up the criteria in terms of s (signature) and r (rank).

Five mutually exclusive possibilities arise:

(i) $\phi = y_1^2 + ...y_n^2$, $(s = r = n).\phi > 0$ for all real values of the variables other than $y_1 = y_2 = ... = y_n = 0$. Hence ϕ is positive definite.

(ii) $\phi = y_1^2 + ... + y_r^2, r < n$ $(s = r < n), \phi \geq 0$ for all real values of the variables other than $y_1 = ... = y_n = 0$. Hence ϕ is positive semi-definite.

(iii) $\phi = -y_1^2 - ... - y_n^2$ $(-s = r = n), \phi < 0$ for all real values of the variables other than $y_1 = ... = y_n = 0$. Hence ϕ is negative definite.

(iv) $\phi = -y_1^2 --..- y_r^2$, $r < n(-s = r < n)$, $\phi \leq 0$ for all real values of the variables other than $y_1 = ...y_n = 0$. Hence ϕ is negative semi-definite.

(v) $\phi = y_1^2 + ... + y_p^2 - y_{p+1}^2 - ..... - y_r^2$, $0 < p < r$ $(|s| \neq r)$. ϕ may have both positive as well as negative values for real values of the variables. Hence ϕ is indefinite.

Since the above five possibilities are mutually exclusive and exhaustive, it follows that the criteria set up above are necessary as well as sufficient. Hence we have the following result:

A real quadratic form is positive definite if and only if $s = r = n$, positive semi-definite if and only if $s = r < n$, negative definite if and only if $-s = r = n$, negative semi-definite if and only if $-s = r < n$, and indefinite if and only if $|s| \neq r$.

- ***Theorem:*** All equivalent quadratic forms have the same value-class.

Proof: Let $\phi =$ x'Ax , ψ =y'By be two equivalent real quadratic forms in n variables. Let U and V be the sets of values which ϕ and ψ can take for all possible real values of the variables (other than those given by $x = 0$ or $y = 0$. We shall show that the sets U, V are identical.

Since ϕ and ψ are equivalent, there exists a real non-singular matrix T such that x =Ty transforms ϕ into ψ.

Let ϕ =p, when $x = a$. The values of ψ when $y = T^{-1}a \; is \left(T^{-1}a\right)' T' A T'\left(T^{-1}a\right) = a' A a = p,$ showing that every member of U is also a member of V. Similarly every member of V is also a member of U. Thus U and V are identical. Hence ϕ and ψ belong to the same value-class.

In particular, it follows that a real quadratic form belong to the same value-class as any of its normal reductions. By virtue of he result proved above , it then follows that a real quadratic form of rank r and signature s in a variables is positive definite if $s = r = n$,positive semi-definite $if \; s = r < n$, negative definite if – s = r = n, negative semi-definite if $-s = r < n,$ and indefinite if $| s |^1 r,$ and conversely.

6.4 Criterion For Positive Definiteness of A Real Quadratic Form

Theorem: The real quadratic form, $\phi =$ x'Ax , in n variables is positive definite if and only if

$$A_1 = a_{11} > 0, \; A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ldots.. A_n = \begin{vmatrix} a_{11} \ldots\ldots a_{1n} \\ . \\ . \quad . \quad . \\ . \quad . \quad . \\ a_{n1} \ldots\ldots a_{nn} \end{vmatrix} > 0$$

Proof:

(i) The conditions are necessary. The proof is by induction on n . The theorem is true for $n = 1$; for if $a_{11}x_1^2$ is positive definite, a$_{11}$ must be positive. Let us now assume the theorem to be true for forms in $n - 1$ variables and show that it also holds for forms in n variables.

Since $\phi = \sum_{j=1}^{n} a_{ij} x_i x_j$ is positive definite, $\phi\left(x_1, \ldots\ldots\ldots, x_{n-1}, 0\right) < 0$.

for all values of $x_1, \ldots.., x_{n-1}$ other than $x_1 = \; \ldots. \; = x_{n-1} = 0$.

Therefore $\phi\left(x_1, \ldots\ldots\ldots, x_{n-1}, 0\right) = \sum_{i,j=1}^{n-1} a_{ij} x_i x_j$, is a positive form in $n - 1$ variables, so that by the inductive hypothesis $A_1 > 0, \ldots., A_{n-1} > 0$.

Since ϕ is positive definite, there exists a non-singular linear transformation $x = Ty$ which transforms ϕ into $y' T y$. Therefore $T' A T = I,$ whence $| T |^2 | A | = 1$. Thus $A_n > 0$.

The proof is now complete by induction.

(ii) The conditions are sufficient. Since $A_1, \ldots, A_n$ are all positive, by theorem, there exists a real non singular linear transformation $x = Ty$ which transforms ϕ into an equivalent quadratic form

$$\psi = A_1 y_1^2 + \frac{A_2}{A_1} y_2^2 + \ldots.. + \frac{A_n}{A_{n-1}} y_n^2 .$$

Since $A_1, \ldots, A_n$ are positive, therefore $A_1, \frac{A_2}{A_1}, \ldots, \frac{A_n}{A_{n-1}}$ are also all positive and ψ is positive definite. Since equivalent quadratic forms have the same value-class, it follows that ϕ is also positive definite.

Corollary: The real quadratic form $f = x' Ax$ is negative definite if and only if $(-1)^r, 1_r, r = 1, 2, \ldots.., n$ are all positive, where A_r stands for the rth order leading principal minor of A.

The proof can be worked out in the same manner as for positive definite forms and is left as an exercise for the reader.

6.5 Classification

A quadratic form is said to be

1. +ve definite, if $f(x_1, x_2, x_3) > 0$ for $x_1 = x_2 = x_3 = 0$
2. –ve definite, if $f(x_1, x_2, x_3) < 0$ for $x_1 = x_2 = x_3 = 0$
3. +ve semi-definite, if $\mathrm{f}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) \geq 0$ for $x_1 = x_2 = x_3 = 0$
4. – ve semi-definite, if $\mathrm{f}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) \leq 0$ for $x_1 = x_2 = x_3 = 0$

Results:

1.

Nature of Matrices	Eigen values
+ ve definite	$\lambda_1 > 0$
- ve definite	$\lambda_i < 0$
+ ve semi-definite	$\lambda_i \geq 0$
- ve semi-definite	$\lambda_1 \leq 0$

2. If all the principal minors of a matrix are positive then it is positive defnite.
3. If the principal minors of a matrix are alternatively negative and positive starting with negative then it is negative definite.

Example: Obtain the matrix corresponding to the given quadratic form

$x_1^2 - 2x_2x_3 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3$

Solution:

The matrix A corresponding to the given quadratic form is

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Here all principal minors are positive so it is a positive definite matrix

Assignment-1

1. Let $p_n(x)=x^n$ for $x\in\mathbb{R}$ and let $V=span\{p_0,p_1,p_2,...\}$. Then

(a.) V is the vector space of all real valued continuous function on $\mathbb{R}$.

(b.) V is a subspace of all real valued continuous function on $\mathbb{R}$.

(c.) $\{p_0,p_1,p_2,...\}$ is a linearly independent set in the vector space of all continuous functions on $\mathbb{R}$.

(d.) Trigonometric functions belong to V.

2. Which of the following are subspaces of the vector space $\mathbb{R}^3$?

(a.) $\{(x,y,z):x+y=0\}$.

(b.) $\{(x,y,z):x-y=0\}$.

(c.) $\{(x,y,z):x+y=1\}$.

(d.) $\{(x,y,z):x-y=1\}$.

3. For arbitrary subspaces U,V and W of a finite dimensional vector space, which of the following hold:

(a.) $U\cap(V+W)\subset U\cap V+U\cap W$

(b.) $U\cap(V+W)\supset U\cap V+U\cap W$

(c.) $(U\cap V)+W\subset(U+W)\cap(V+W)$

(d.) $(U\cap V)+W\supset(U+W)\cap(V+W)$

4. Let V denote a vector space over a field F and with a basis $B=\{e_1,e_2,...,e_n\}$. Let $x_1,x_2,...,x_n\in F$.
Let $C=\{x_1e_1,x_1e_1+x_2e_2,...,x_1e_1+x_2e_2+...+x_ne_n\}$. Then

(a.) C is a linearly independent set implies that $x_i\neq 0$ for every $i=1,2,...,n$

(b.) $x_i\neq 0$ for every $i=1,2,...,n$ implies that C is a linearly independent set.

(c.) The linear span of C is V implies that $x_i\neq 0$ for every $i=1,2,...,n$

(d.) $x_i\neq 0$ for every $i=1,2,...,n$ implies that the linear span of C is V

5. Consider the following row vectors :

$$a_1=(1,1,0,1,0,0),a_2=(1,1,0,0,1,0),$$
$$a_3=(1,1,0,0,0,1),\ a_4=(1,0,1,1,0,0),$$
$$a_5=(1,0,1,0,1,0),a_6=(1,0,1,0,0,1),$$

The dimension of the vector space spanned by these row vectors is

(a.) 6.
(b.) 5.
(c.) 4.
(d.) 3.

6. Let $\{v_1,\cdots,v_n\}$ be a linearly independent subset of a vector space V where $n\geq 4$. Set $w_{ij}=v_i-v_j$. Let W be the span of $\{w_{ij}\,|\,1\leq i,j\leq n\}$. Then

(a.) $\{w_{ij}\,|\,1\leq i<j\leq n\}$ spans W.

(b.) $\{w_{ij}\,|\,1\leq i<j\leq n\}$ is a linearly independent subset of W.

(c.) $\{w_{ij}\,|\,1\leq i\leq n-1, j=i+1\}$ spans W.

(d.) $\dim W=n$.

7. Let V be a 3-dimensional vector space over the field $\mathbb{F}_3=\frac{\mathbb{Z}}{3\mathbb{Z}}$ of 3 elements. The number of distinct 1-dimensional subspaces of V is

(a.) 13
(b.) 26
(c.) 9
(d.) 15

8. Let n be an integer, $n\geq 3$,. and let $u_1,u_2,...,u_n$ be n linearly independent elements in a vector space over $\mathbb{R}$. Set $u_0=0$ and $u_{n+1}=u_1$. Define $v_i=u_i+u_{i+1}$ and $w_i=u_{i-1}+u_i$ for $i=1,2,...,n$. Then

(a.) $v_1,v_2,...,v_n$ are linearly independent, if $n=2010$

(b.) $v_1,v_2,...,v_n$ are linearly independent, if $n=2011$

(c.) $w_1,w_2,...,w_n$ are linearly independent, if $n=2010$

(d.) $w_1,w_2,...,w_n$ are linearly independent, if $n=2011$

9. The dimension of the vector space of all symmetric matrices $A=\left(a_{jk}\right)$ of order $n\times n(n\geq 2)$ with real entries, $a_{11}=0$ and trace zero is

(a.) $\frac{\left(n^2+n-4\right)}{2}$

(b.) $\frac{\left(n^2-n+4\right)}{2}$

(c.) $\frac{\left(n^2+n-3\right)}{2}$

(d.) $\frac{\left(n^2-n+3\right)}{2}$

10. Let V_1, V_2 be subspaces of a vector space V, which of the following is necessarily a subspace of V?

(a.) $V_1\cap V_2$

(b.) $V_1\cup V_2$

(c.) $V_1+V_2=\left\{x+y:x\in V_1, y\in V_2\right\}$

(d.) $V_1\setminus V_2=\left\{x\in V_1 \text{ and } x\notin V_2\right\}$

11. Let $S=\left\{A:A\left[a_{ij}\right]_{5\times 5}, a_{ij}=0 \text{ or } 1\ \forall i, j, \sum_{j} a_{ij}=1\forall i \text{ and } \sum_{j} a_{ij}=1\forall j\right\}$.

Then the number of elements is S is

(a.) 5^2

(b.) 5^5

(c.) 5!

(d.) 55

12. The dimension of the vector space of all symmetric matrices of order $n\times n(n\geq 2)$ with real entries and trace equal to zero is

(a.) $\frac{\left(n^2-n\right)}{2}-1$

(b.) $\frac{\left(n^2-2n\right)}{2}-1$

(c.) $\frac{\left(n^2+n\right)}{2}-1$

(d.) $\frac{\left(n^2+2n\right)}{2}-1$

13. Let M be the vector space of all 3 × 3 real matrices and let

$$A=\begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}$$

Which of the following are subspaces of M?

(a.) $\{X\in M: XA=AX\}$

(b.) $\{X\in M: X+A=A+X\}$

(c.) $\{X\in M: \text{trace}\,(AX)=0\}$

(d.) $\{X\in M: \det\,(AX)=0\}$

14. Let $W=\{p(B): p$ is a polynomial with real coefficients$\}$, where $B=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$

The dimension d of the vector space W satisfies

(a.) $4\leq d\leq 6$

(b.) $6\leq d\leq 9$

(c.) $3\leq d\leq 8$

(d.) $3\leq d\leq 4$

15. Let V be a real vector space and let $\{x_1, x_2, x_3\}$ be a basis for V. Then

(a.) $\{x_1+x_2, x_2, x_3\}$ is a basis for V

(b.) The dimension of V is 3

(c.) $\{x_1-x_2, x_2-x_3, x_1-x_3\}$ is a basis for V

(d.) None of these

16. Let V be the set of all real $n\times n$ matrices $A=\left(a_{ij}\right)$ with the property that $a_{ij}=-a_{ji}$ for all $i, j=1, 2, \ldots, n$. Then

(a.) V is a vector space of dimension n^2-n

(b.) For every A in V, $a_{ii}=0$ for all $i=1, 2, \ldots, n$

(c.) V consists of only diagonal matrices

(d.) V is a vector space of dimension $\frac{n^2-n}{2}$

17. Let $V=\{(x_1,\dots,x_{100})\in\mathbb{R}^{100}: x_1=\dots=x_{50}$ and $x_{51}+x_{52}+\dots+x_{100}=0\}$. Then

(a.) $\dim V=98$

(b.) $\dim V=51$

(c.) $\dim V=49$

(d.) $\dim V=50$

18. Let $P(3)=\{a_0+a_1x+a_2x^2+a_3x^3 \mid a_i\in\mathbb{R},\ i=0,1,2,3\}$

Under the standard operation of addition (+) and scalar multiplication (.)P(3) is

(a.) Not a vector space

(b.) A vector space of infinite dimension

(c.) A vector space of dimension 3

(d.) A vector space of dimension 4

19. Let $V=\{(x_1,\dots;x_{100})\in R^{100} \mid x_1=2x_2=3x_3$

and $x_{51}-x_{52}-\dots-x_{100}=0\}$

Then

(a.) dim V = 98

(b.) dim V = 49

(c.) dim V = 99

(d.) dim V = 97

20. Consider the linear differential equation

$$y^{(3)}+3xy^{(2)}+4y^{(1)}+2x^2y=\sin x,\left(x\in[0,1]\right)$$

.

Then the set of solutions of the above equation

(a.) Is a linear space of infinite dimension

(b.) Is a linear space of dimension 3

(c.) Is not a linear space

(d.) Is a linear space of dimension less than 3

21. Let $\{e_1,e_2,e_3\}$ be a basis of a vector space V over R. Consider the following sets:

$A=\{e_2,e_1+e_2,e_1+e_2+e_3\}$

$B=\{e_1,e_1+e_2,e_1+e_2+e_3\}$

$C=\{2e_1,3e_1+e_3,6e_1+3e_2+e_3\}$

(a.) A and B are bases of V

(b.) A and C are bases of V

(c.) B and C are bases of V

(d.) Only B is basis of V

22. Let V be the vector space of all 5×5 real skew-symmetric matrices. Then the dimension of V is

(a.) 20

(b.) 15

(c.) 10

(d.) 5

23. U is a subset of $\mathbb{R}^4$ given by $U=\{(x_1,x_2,x_3,x_4)\mid(x_1-x_2+x_3=0=x_1+x_2+x\ )\}$ then

(a.) U is not a subspace of $\mathbb{R}^4$

(b.) U is a subspace of $\mathbb{R}^4$ of dimension 1

(c.) U is a subspace of $\mathbb{R}^4$ of dimension 2

(d.) U is a subspace of $\mathbb{R}^4$ of dimension 3

24. Let $S_2=\{x_1,x_2,\dots,x_m\}$ and $T=\{y_1,\dots;y_n\}$ be subsets of the vector space V. Then

(a.) If S and T are both linearly independent then $m=n$

(b.) If S is a basis for V and if T spans V then $m\ge n$

(c.) If S is a basis for V and if T is linearly independent, then $m\ge n$

(d.) If S is linearly independent and if T spans V, then $m\le n$

25. Let $S=\{(0,1,\alpha),(\alpha,1,0),(1,\alpha,1)\}$. Then S is a basis for $\mathbb{R}^3$ if and only if

(a.) $\alpha\ne 0$

(b.) $\alpha\ne 1$

(c.) $\alpha\ne 0$ and $\alpha^2\ne 2$

(d.) $-1\le a\le 1$

26. Let S be the set of all $n\times n$ matrices over $\mathbb{R}$ with zero trace. Then

(a.) S is not a vector space

(b.) S is a vector space of dimension $n-1$

(c.) S, together with the identity matrix, form a vector space

(d.) S is a vector space and it has a basis consisting of n^2-1 matrices

27. Let A and B be $n \times n$ matrices. Then

(a.) Every column of AB can be expressed as a linear combination of columns of A

(b.) Every column of AB can be expressed as a linear combination of rows of A

(c.) Every column of AB can be expressed as a linear combination of column of B

(d.) Every row of AB can be expressed as a linear combination of rows of A

28. Let V be the vector space of real polynomials of degree not exceeding 2. Let $f(x)=x-1, g(x)=x+1, h(x)=x^2-1, j(x)=x^2+1$.

Then the set $\{f,g,h,j\}$ is

(a.) Linearly independent

(b.) Linearly dependent because $fg=h$

(c.) Linearly dependent because $f+g-h=0$

(d.) Linearly dependent because $f-g-h+j=0$

29. Let W be the subspace of $\mathbb{R}^2$ spanned by $(1,2)$. Which of the following pairs represent the same element of the quotient space $\frac{\mathbb{R}^2}{W}$

(a.) $W, \left(\frac{1}{2}, 3\right)+W$

(b.) $\left(1, \frac{1}{2}\right)+W, \left(6 \frac{21}{2}\right)+W$

(c.) $\left(1, \frac{1}{2}\right)+W, \left(2 \frac{3}{2}\right)+W$

(d.) $\left(\sqrt{2}, 1\right)+W, \left(\sqrt{2}, 2\right)+W$

30. Suppose V_1, V_2, V_3, V_4 are linearly independent vectors of a real vector space. Consider the two sets of vectors

$$S_1=\{V_1+V_2, V_1+V_3, V_1+V_4\}$$

$$S_2=\{V_1+V_2, V_1+V_3, V_1+V_4, V_4+V_1\}$$

$$S_2=\{V_1+V_2, V_1+V_3, V_1+V_4, V_4+V_1\}$$

Which of the following is true?

(a.) Both S_1 and S_2 are linearly independent

(b.) S_2 is linearly independent but not S_1

(c.) S_1 is linearly independent but not S_2

(d.) Neither S_1 nor S_2 is linearly independent

31. Which of the following is not a subspace of $\mathbb{R}^3$?

(a.) $\{(a,b,c) \mid a,b,c \text{ rational}\}$

(b.) $\{(0,0,0)\}$

(c.) $\{(a, a+b, -a+2b) \mid a,b \text{ real}\}$

(d.) $\{(a, a-b, b) \mid a,b \text{ real}\}$

32. Let V denote the vector space $C^5[a,b]$ over $\mathbb{R}$ and $W=\left\{f \in V : \frac{d^4 f}{dt^4}+2\frac{d^2 f}{dt^2}-f=0\right\}$.

Then

(a.) $\dim(V)=\infty$ and $\dim(W)=\infty$

(b.) $\dim(V)=\infty$ and $\dim(W)=4$

(c.) $\dim(V)=6$ and $\dim(W)=5$

(d.) $\dim(V)=5$ and $\dim(W)=4$

33. Let F_3 be the field of 3 elements and let $F_3 \times F_3$ be the vector space over F_3. The number of distinct linearly dependent sets of the from $\{u,v\}$, where $u,v \in F_3 \times F_3 / \{(0,0)\}$ and $u \neq v$ is __

34. Let M be the space of all 4 × 3 matrices with entries in the finite field of three elements. Then the number of matrices of rank three in M is

(a.) $(3^4-3)(3^4-3^2)(3^4-3^3)$

(b.) $(3^4-1)(3^4-2)(3^4-3)$

(c.) $(3^4-1)(3^4-3)(3^4-3^2)$

(d.) $3^4(3^4-1)(3^4-2)$

35. The dimension of the vector space $V=\{A=[a_{ij}]_{n\times n} \mid a_{ij} \in \mathbb{C}, a_{ij}=-a_{ji}\}$ over field $\mathbb{R}$ is

(a.) n^2

(b.) n^2-1

(c.) n^2-n

(d.) $\frac{n^2}{2}$

36. Consider the subspace $W=\{[a_{ij}]: a_{ij}=0 \text{ if } i \text{ is even}\}$ of all 10×10 real matrices. Then the dimension of W is

(a.) 25
(b.) 50
(c.) 75
(d.) 100

37. Let $M=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, where $0<\theta<\frac{\pi}{2}$. Let $V=\{u\in\mathbb{R}^3 : M u^t = u^t\}$. Then the dimension of V is

(a.) 0
(b.) 1
(c.) 2
(d.) 3

38. Let $A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ x & y & z \end{bmatrix}$ and let $V=\{(x,y,z)\in\mathbb{R}^3 : \det(A)=0\}$. Then the dimension of V equals

(a.) 0
(b.) 1
(c.) 2
(d.) 3

39. A basis of $V=\{(x,y,z,w)\in\mathbb{R}^4 : x+y-z=0,$ $y+z+w=0, 2x+y-3z-w=0\}$

(a.) $\{(1,1,-1,0),(0,1,1,1),\{(2,1,-3,1)\}$
(b.) $\{(1,-1,0,1)\}$
(c.) $\{(1,0,1,-1)\}$
(d.) $\{(1,-1,0,1),(1,0,1-1)\}$

40. The dimension of the subspace $\{(x_1,x_2,x_3,x_4,x_5): 3x_1-x_2+x_3=0\}$ of $\mathbb{R}^5$ is

(a.) 1
(b.) 2
(c.) 3
(d.) 4

41. Let V be the vector space of all real polynomials. Consider the subspace W spanned by t^2+t+2, t^2+2t+5, $5t^2+3t+4$ and $2t^2+2t+4$. Then the dimension of W is

(a.) 4
(b.) 3
(c.) 2
(d.) 1

42. The set of all $x\in\mathbb{R}$ for which the vectors $(1,x,0)$, $(0,x^2,1)$ and $(0,1,x)$ are linearly independent in R^3 is

(a.) $\{x\in\mathbb{R} : x=0\}$
(b.) $\{x\in\mathbb{R} : x\neq 0\}$
(c.) $\{x\in\mathbb{R} : x\neq 1\}$
(d.) $\{x\in\mathbb{R} : x\neq -1\}$

43. Let S and T be two subspace of $\mathbb{R}^{24}$ such that dim $(S)=19$ and dim $(T)=17$. Then, the

(a.) Smallest possible value of $\dim(S\cap T)$ is 2
(b.) Largest possible value of $\dim(S\cap T)$ is 18
(c.) Smallest possible value of $\dim(S+T)$ is 19
(d.) Largest possible value of $\dim(S+T)$ is 22

44. Let $v_1=(1,2,0,3,0), v_2=(1,2,-1,-1,0)$, $v_3=(0,0,1,4,0)$, $v_4=(2,4,1,10,1)$ and $v_5(0,0,0,0,1)$. The dimension of the linear span of (v_1,v_2,v_3,v_4,v_5) is

(a.) 2
(b.) 3
(c.) 4
(d.) 5

45. The set $V=\{(x,y)\in\mathbb{R}^2 : xy\geq 0\}$ is

(a.) A vector subspace of $\mathbb{R}^2$
(b.) Not a vector subspace of $\mathbb{R}^2$ since every element does not have an inverse in V

(c.) Not a vector subspace of $\mathbb{R}^2$ since it is not closed under scalar multiplication

(d.) Not a vector subspace of $\mathbb{R}^2$ since it is not closed under vector addition

46. Let $\{v_1, v_2, v_3, v_4\}$ be a basis of $\mathbb{R}^4$ and $v = a_1v_1 + a_2v_2 + a_3v_3$ where $a_i \in \mathbb{R}$, $i = 1, 2, 3, 4.$
Then $\{v_1 - v, v_2 - v, v_3 - v, v_4 - v\}$ is a basis of $\mathbb{R}^4$ if and only if

(a.) $a_1 = a_2 = a_3 = a_4$

(b.) $a_1a_2a_3a_4 = -1$

(c.) $a_1 + a_2 + a_3 + a_4 \neq 0$

(d.) $a_1 + a_2 + a_3 + a_4 = 0$

47. The dimension of the vector space of all 3×3 real symmetric matrices is

(a.) 3

(b.) 9

(c.) 6

(d.) 4

48. Let W be the space spanned by $f = \sin x$ and $g = \cos x$. Then for any real value of $\theta, f_1 = \sin(x+\theta)$ and $g_1 = \cos(x+\theta)$.

(a.) Are vectors in W

(b.) Are linearly independent

(c.) Do not form a basis for W

(d.) Form a basis for W

49. Let W be an m-dimensional subspace of an n-dimensional vector space V, where $m < n$. Then the dimension of V/W is

(a.) $\frac{n}{m}$

(b.) $n - m$

(c.) $n + m$

(d.) 0

50. Consider the set

$$V = \{(x_1 - x_2 + x_3, x_1 + x_2 - x_3) : (x_1, x_2, x_3) \in \mathbb{R}^3\}$$

Then

(a.) V is not a vector subspace of $\mathbb{R}^2$

(b.) V is a vector subspace of $\mathbb{R}^2$ of dimension 0

(c.) V is a vector subspace of $\mathbb{R}^2$ of dimension 1

(d.) $V = \mathbb{R}^2$

ASSIGNMENT-2

1. Let V be the space of twice differentiable functions on $\mathbb{R}$ satisfying $f'' - 2f' + f = 0$. Define $T: V \to \mathbb{R}^2$ by $T(f) = (f'(0), f(0))$. Then T is

(a.) one-to-one and onto.

(b.) one-to-one but not onto.

(c.) onto but not one-to-one.

(d.) neither one-to-one nor onto.

2. Given a 4×4 real matrix A, let $T:\mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by $Tv = Av$, where we think of $\mathbb{R}^4$ as the set of real 4×1 matrices. For which choices of A given below, do Image(T) and Image (T^2) have respective dimensions 2 and 1? (* denotes a nonzero entry)

(a.) $A = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b.) $A = \begin{bmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix}$

(c.) $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{bmatrix}$

(d.) $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$

3. Which of the following is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$?

A. $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ x+y \end{pmatrix}$

B. $g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix}$

C. $h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z-x \\ x+y \end{pmatrix}$

(a.) only f.

(b.) only g.

(c.) only h.

(d.) all the transformations f, g and h.

4. Consider non-zero vector spaces V_1, V_2, V_3, V_4 and linear transformations $\phi_1 : V_1 \to V_2$, $\phi_2 : V_2 \to V_3$, $\phi_3 : V_3 \to V_4$ such that $\ker(\phi_1) = \{0\}$, Range $(\phi_1) = \ker(\phi_2)$, Range $(\phi_2) = \ker(\phi_3)$, Range $(\phi_3) = V_4$. Then

(a.) $\sum_{i=1}^{4}(-1)^i \dim V_i = 0$.

(b.) $\sum_{i=2}^{4}(-1)^i \dim V_i > 0$.

(c.) $\sum_{i=1}^{4}(-1)^i \dim V_i < 0$.

(d.) $\sum_{i=1}^{4}(-1)^i \dim V_i \neq 0$.

5. Let $M_n(K)$ denote the space of all $n\times n$ matrices with entries from afield K. Fix a non-singular matrix $A = (A_{ij}) \in M_n(K)$ and consider the linear map $T: M_n(K) \to M_n(K)$ given by: $T(X) = AX$. Then

(a.) Trace $(T) = n\sum_{i=1}^{n} A_{ii}$

(b.) Trace $(T) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}$

(c.) Rank of T is n^2

(d.) T is non-singular

6. Let $M_{m\times n}(\mathbb{R})$ be the set of all $m\times n$ matrices with real entries. Which of the following statement is correct?

(a.) There exists $A\in M_{2\times 5}(\mathbb{R})$ such that the dimension of the null space of A is 2.

(b.) There exists $A\in M_{2\times 5}(\mathbb{R})$ such that the dimension of the null space of A is 0.

(c.) There exists $A\in M_{2\times 5}(\mathbb{R})$ and $B\in M_{5\times 2}(\mathbb{R})$ such that AB is the 2×2 identity matrix

(d.) There exists $A\in M_{2\times 5}(\mathbb{R})$ whose null space is $\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{R}^5 : x_1 = x_2, x_3 = x_4 = x_5\}$

7. Let V be the vector space of polynomials over R of degree less than or equal to n. For $p(x)=a_0+a_1x+...+a_nx^n$ in V, define a linear transformation $T:V\to V$ by $(Tp)(x)=a_0+a_1x+...+a_nx^2-....+(-1)^n a_nx^n$. Then which of the following are correct?

(a.) T is one-to-one

(b.) T is onto

(c.) T is invertible

(d.) det $T=0$

8. A linear transformation T rotates each vector in $\mathbb{R}^2$ clockwise through 90°. The matrix T relative to the standard ordered basis $\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)$ is

(a.) $\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$

(b.) $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

(c.) $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

(d.) $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

9. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a linear transformation. Which of the following statements implies that T is bijective?

(a.) Nullity $(T)=n$

(b.) Rank $(T)=$ Nullity $(T)=n$

(c.) Rank (T) + Nullity $(T)=n$

(d.) Rank (T) − Nullity $(T)=n$

10. Let n be a positive integer and let $M_n(\mathbb{R})$ denote the space of all $n\times n$ real matrices. If $T:M_n(\mathbb{R})\to M_n(\mathbb{R})$ is a linear transformation such that $T(A)=0$ whenever $A\in M_n(\mathbb{R})$ is symmetric or skew-symmetric, then the rank of T is

(a.) $\dfrac{n(n+1)}{2}$

(b.) $\dfrac{n(n-1)}{2}$

(c.) n

(d.) 0

11. Let $S:\mathbb{R}^3\to\mathbb{R}^4$ and $T:\mathbb{R}^4\to\mathbb{R}^3$ be linear transformations such that $T\circ S$ is the identity map of $\mathbb{R}^3$. Then

(a.) $S\circ T$ is the identity map of $\mathbb{R}^4$

(b.) $S\circ T$ is one-one, but not onto

(c.) $S\circ T$ is onto, but not one-one

(d.) $S\circ T$ is neither one-one nor onto.

12. Let $a,b,c,d\in\mathbb{R}$ and let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the linear transformation defined by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}ax+by\\cx+dy\end{bmatrix}$ for $\begin{bmatrix}x\\y\end{bmatrix}\in\mathbb{R}^2$, Let $S:\mathbb{C}\to\mathbb{C}$ be the corresponding map defined by $S(x+iy)=(ax+by)+i(cx+dy)$ for $x,y\in\mathbb{R}$. Then

(a.) S is always $\mathbb{C}$-linear, that is $S(z_1+z_2)=S(z_1)+S(z_2)$ for all $z_1,z_2\in\mathbb{C}$ and $S(\alpha z)=\alpha S(z)$ for all $\alpha\in\mathbb{C}$ and $z\in\mathbb{C}$.

(b.) S is $\mathbb{C}$-linear if $b=-c$ and $d=a$

(c.) S is $\mathbb{C}$-linear only if $b=-c$ and $d=a$

(d.) S is $\mathbb{C}$-linear if and only if T is the identity transformation.

13. Let n be a positive integer and V be an $(n+1)$-dimensional vector space over $\mathbb{R}$. If $\{e_1, e_2, \ldots, e_{n+1}\}$ is a basis of V and $T: V \to V$ is the linear transformation satisfying $T(e_i) = e_{i+1}$ for $i = 1, 2, \ldots, n$ and $T(e_{n+1}) = 0$. Then

(a.) Trace of T is non zero

(b.) Rank of T is n

(c.) Nullity of T is 1

(d.) $T^n = T \circ T \circ \ldots \circ T$ (n times) is the zero map.

14. Let V be the vector space of all real polynomials of degree at most 3. Define $S: V \to V$ by $S(p(x)) = Q(x), \forall p(x) \in V$, Where $Q(x) = p(x+1)$, Then the matrix of S in the basis $\{1, x, x^2, x^3\}$, considered as column vectors, is given by:

(a.) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(b.) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c.) $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$

(d.) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

15. For a positive integer n, let P_n denote the space of all polynomials $p(x)$ with coefficients in $\mathbb{R}$ such that deg $p(x) \le n$, and let B_n denote the standard basis of P_n given by $B_n = \{1, x, x^2, \ldots, x^n\}$. If $T: P_3 \to P_4$ is the linear transformation defined by $T(p(x)) = x^2 p'(x) + \int_0^x p(t)\,dt$ and $A = [a_{ij}]$ is the 5×4 matrix of T with respect to standard bases B_3 and B_4, then

(a.) $a_{32} = \frac{3}{2}$ and $a_{33} = \frac{7}{3}$

(b.) $a_{32} = \frac{3}{2}$ and $a_{33} = 0$

(c.) $a_{32} = 0$ and $a_{33} = \frac{7}{3}$

(d.) $a_{32} = 0$ and $a_{33} = 0$

16. Consider the linear transformation $T: \mathbb{R}^7 \to \mathbb{R}^7$ defined by

$T(x_1, x_2, \ldots, x_6, x_7) = (x_7, x_6, \ldots, x_2, x_1)$

Which of the following statements are true?

(a.) The determinant of T is 1

(b.) There is a basis of $\mathbb{R}^7$ with respect to which T is a diagonal matrix

(c.) $T^7 = I$

(d.) The smallest n such that $T^n = I$ is even

17. Let W be the vector space of all real polynomials of degree at most 3. Define $T: W \to W$ by $(T(p(x)) = p'(x)$ where p' is the derivative of p. The matrix of T in the basis $\{1, x, x^2, x^3\}$, considered as column vectors, is given by

(a.) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(b.) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$

(c.) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(d.) $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

18. Let x, y be linearly independent vectors in $\mathbb{R}^2$ suppose $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear-transformation such that $Ty = ax$ and $Tx = 0$. Then with respect to some basis in $\mathbb{R}^2, T$ of the form

(a.) $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a > 0$

(b.) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b > 0, a \neq b$

(c.) $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$

(d.) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

19. Let V be the space of all linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$ under usual addition and scalar multiplication. Then

(a.) V is a vector space of dimension 5

(b.) V is a vector space of dimension 6

(c.) V is a vector of dimension 8

(d.) V is a vector space of dimension 9

20. Let $A:\mathbb{R}^6 \to \mathbb{R}^5$ and $B:\mathbb{R}^5 \to \mathbb{R}^7$ be two linear transformations. Then which of the following can be true?

(a.) A and B are none one-one

(b.) A is one-one and B is not one-one

(c.) A is onto and B is one-one

(d.) A and B both are onto.

21. The transformation $(x, y, z) \to (x+y, y+z): \mathbb{R}^3 \to \mathbb{R}^2$ is

(a.) Linear and has zero kernel

(b.) Linear and has a proper subspace as kernel

(c.) Neither linear nor 1–1

(d.) Neither linear nor onto

22. Let $T:\mathbb{R}^3 \to W$ be the orthogonal projection of $\mathbb{R}^3$ on to the xz plane W. Then

(a.) $T(x, y, z) = (x+y, 0, y+z)$

(b.) $T(x, y, z) = (x-y, 0, y-z)$

(c.) $T(x, y, z) = (x+y+z, 0, z)$

(d.) $T(x, y, z) = (x, 0, z)$

23. $\{v_1, v_2, v_3\}$ is a basis of $V = \mathbb{R}^3$ $\mathbb{R}^3$ and linear transformation $T:V \to V$ is define by
$T(v_1) = v_1 + v_2, T(v_2) = v_2 + v_3, T(v_3) = v_3 + v_1,$
then

(a.) T is 1-1

(b.) $T(v_1 + v_2 + v_3) = 0$

(c.) $T(v_1 + 2v_2 + v_3) = T(v_2 + 2v_3)$

(d.) $T(v_1 - v_3) = v_1 - v_3$

24. For the standard basi $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{R}^3$, a linea transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ has th matrix representation $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$. Th image under T of (2, 1, 2) is

(a.) (11, 0, -1)

(b.) (11, 3, -5)

(c.) (7, 3, -1)

(d.) None these

25. For a linear transformation $T:\mathbb{R}^{10} \to \mathbb{R}^6$, th kernel is having dimension 5. Then th dimension of the range of T is

(a.) Five

(b.) Six

(c.) Four

(d.) Two

26. Consider the 3×3 matrix $T = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

Which of the following is false?

(a.) There is a non-zero vector which is not the image of T

(b.) There is a non-zero vector which is in t kernel of T

(c.) There is a non-zero vector which is not the kernel of T

(d.) The matrix is invertible

27. Let V be the vector space of 2×2 matrices over $\mathbb{R}$. Which of the following is/are not linear transformations?

A. $T([a_{ij}]) = a_{11}$

B. $T(A) = A+1$

C. $T(A) = \text{Trace } A$

D. $T([a_{ij}]) = a_{11}+a_{12}+a_{21}+a_{22}$

(a.) A and B

(b.) Only B

(c.) Only C

(d.) Only D

28. Let V be the real vector space of real polynomials of degree < 3 and let $T: V \to V$ be the linear transformation defined by $P(t)$ and $Q(t)$ where $Q(t) = p(at+b)$. Then the matrix of T with respect to the basis $1, t, t^2$ of V is:

(a.) $\begin{pmatrix} 1 & b & b^2 \\ 0 & a & 2ab \\ 0 & 0 & a^2 \end{pmatrix}$

(b.) $\begin{pmatrix} 0 & a & a^2 \\ 0 & b & 2ab \\ 0 & 0 & b^2 \end{pmatrix}$

(c.) $\begin{pmatrix} b & b & b^2 \\ a & a & 0 \\ 0 & b & a^2 \end{pmatrix}$

(d.) $\begin{pmatrix} a & a & a^2 \\ b & b & 0 \\ 0 & a & b^2 \end{pmatrix}$

29. Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map defined by $T(x,y,z,w) = (x+z, 2x+y+3z, 2y+2z, w)$. Then the rank of T is equal to ________

30. Let $T_1, T_2 : \mathbb{R}^5 \to \mathbb{R}^3$ be linear transformations such that rank $(T_1) = 3$ and nullity $(T_2) = 3$. Let $T_3 : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T_3 \circ T_1 = T_2$. then rank (T_3) is ________

31. Let M be the real vector space of 2×3 matrices with real entries. Let $T: M \to M$ be defined by $T\left(\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}\right) = \begin{bmatrix} -x_6 & x_4 & x_1 \\ x_3 & x_5 & x_2 \end{bmatrix}$. Then determinant of T is ________

32. Let linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x_1, x_2) = (x_1, x_1+x_2, x_2)$. Then the nullity of T is

(a.) 0

(b.) 1

(c.) 2

(d.) 3

33. Let $P_3 = \{p(x) \mid p(x)$ be a polynomial with real coefficient and degree at-most 3 $\}$ and $T: P_3 \to P_3$ be the map given by $T(p(x)) = \int_1^x p'(t)\,dt$. If the matrix of T relative to the standard bases $B_1 = B_2 = \{1, x, x^2, x^3\}$ is M and M' denotes the transpose of the matrix M, then $M + M'$ is

(a.) $\begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(b.) $\begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 2 & -1 \end{bmatrix}$

(c.) $\begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$

(d.) $\begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{bmatrix}$

34. If the nullity of the matrix $\begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{bmatrix}$ is 1, then the value of k is

(a.) -1

(b.) 0

(c.) 1

(d.) 2

35. Let $T: P_3[0,1] \to P_2[0,1]$ be defined by $T(p(x)) = p''(x) + p'(x)$. Then the matrix representation of T with respect to the bases $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$ of $P_3[0,1]$ and $P_2[0,1]$ respectively is

(a.) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 6 & 3 \end{bmatrix}$

(b.) $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(c.) $\begin{bmatrix} 0 & 2 & 1 & 0 \\ 6 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$

(d.) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 6 & 0 \end{bmatrix}$

36. Consider the basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$; where $u_1 = (1,0,0)$, $u_2 = (1,1,0)$, $u_3 = (1,1,1)$. Let $\{f_1, f_2, f_3\}$ be the dual basis of $\{u_1, u_2, u_3\}$ and f be a linear functional defined by $f(a,b,c) = a+b+c$, $(a,b,c) \in \mathbf{R}^3$. If $f = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$, then $(\alpha_1, \alpha_2, \alpha_3)$ is

(a.) $(1,2,3)$

(b.) $(1,3,2)$

(c.) $(2,3,1)$

(d.) $(3,2,1)$

37. Let V be the column space of the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$. Then the orthogonal projection of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ on V is

(a.) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(b.) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c.) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(d.) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

38. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3, 3x_1 + 4x_2 + x_3, 2x_1 + x_2 - x_3)$

I. The dimension of the range space of T^2 is

(a.) 0

(b.) 1

(c.) 2

(d.) 3

II. The dimension of the null space of T^3 is

(a.) 0

(b.) 1

(c.) 2

(d.) 3

36. Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map satisfying $T(e_1) = e_2, T(e_2) = e_3, T(e_3) = 0, T(e_4) = e_3$ Where $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$. Then

(a) T is idempotent

(b) T is invertible

(c) Rank $T = 3$

(d) T is nilpotent

37. For any $n \in N$, let P_n denote the vector space of all polynomials with real coefficients and of degree at most n. Define $T : P_n \to P_{n+1}$ by

$$T(p)(x) = p'(x) - \int_0^x p(t)\,dt$$

Then the dimension of the null space of T is

(a.) 0

(b.) 1

(c.) n

(d.) $n+1$

38. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T((x,y,z)) = (x+y-z, x+y+z, y-z)$. Then the matrix of the linear transformation T with respect to the ordered basis $B = \{(0,1,0),(0,0,1),(1,0,0)\}$ of $\mathbb{R}^3$ is

(a.) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

(b.) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

(c.) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

(d.) $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

40. Let the linear transformations S and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$S(x,y,z) = (2x, 4x-y, 2x+3y-z)$$

$$T(x,y,z) = (x\cos\theta - y\sin\theta, \sin\theta + y\cos\theta, z)$$

where $0 < \theta < \pi/2$. Then

(a.) S is one to one but not T

(b.) T is one to one but not S

(c.) Both S and T are one to one

(d.) Neither S nor T is one to one

42. Consider the vector space $\mathbb{R}^3$ and the maps $f, g : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x,y,z) = (x,|y|,z)$ and $f(x,y,z) = (x+1, y-1, z)$. Then

(a.) Both f and g are linear

(b.) Neither f nor g is linear

(c.) g is linear but not f

(d.) f is linear but not g

43. Let S and T be two linear operators on $\mathbb{R}^3$ defined by $S(x,y,z) = (x, x+y, x-y-z)$ $T(x,y,z) = (x+2z, y-z, x+y+z)$. Then

(a.) S is invertible but not T

(b.) T is invertible but not S

(c.) Both S and T are invertible

(d.) Neither S nor T is invertible

44. Let V, W and X be three finite dimensional vector spaces such that $\dim V = \dim X$. Suppose $S : V \to W$ and $T : W \to X$ are two linear maps such that $ToS : V \to X$ is injective. Then

(a.) S and T are surjective

(b.) S is surjective and T is injective

(c.) S and T are injective

(d.) S is injective and T is surjective

45. Let $\mathbb{R}^{2\times 2}$ be the real vector space of all 2×2 real matrices. For $Q = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$, define a linear transformation T on $\mathbb{R}^{2\times 2}$ as $T(P) = QP$. Then the rank of T is

(a.) 1

(b.) 2

(c.) 3

(d.) 4

46. Let T an arbitrary linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ which is not one-one.

Then

(a.) Rank T > 0

(b.) Rank T = n

(c.) Rank T < n

(d.) Rank T = n −1

47. Let T be a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z)=(x+y,y-z)$. Then the matrix of T with respect to the ordered bases $\{(1,1,1),(1,-1,0),(0,1,0)\}$ and $\{(1,1),(1,0)\}$ is

(a.) $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

(b.) $\begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

(c.) $\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$

(d.) $\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$

48. Choose the correct matching from A, B, C and D for the transformation T_1, T_2 and T_3 (mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$) as defined in Group 1 with the statements given in Group 2.

Group 1

P $T_1(x,y)=(x,x,0)$

Q $T_2(x,y)=(x,x+y,y)$

R $T_3(x,y)=(x,x+1,y)$

Group 2

1. Linear transformation of rank 2
2. Not a linear transformation
3. Linear transformation

(a.) P–3, Q – 1, R–2

(b.) P–1, Q–2, R–3

(c.) P–3, Q–2, R–1

(d.) P–1, Q–3, R–2

49. Consider the basis $S=\{v_1,v_2,v_3\}$ for R^3 where $v_1=(1,1,1), V_2=(1,1,0)$, $v_3=(1,0,0)$ and let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $Tv_1=(1,0), Tv_2=(2,-1), Tv_3=(4,3)$. Then $T(2,-3,5)$ is

(a.) $(-1,5)$

(b.) $(3,4)$

(c.) $(0,0)$

(d.) $(9,23)$

ASSIGNMENT-3

1. The row space of a 20×50 matrix A has dimension 13. What is the dimension of the space of solutions of $Ax=0$?

(a.) 7

(b.) 13

(c.) 33

(d.) 37

2. Let A be an $m\times n$ matrix of rank n with real entries. Choose the correct statement.

(a.) $Ax=b$ has a solution for any b.

(b.) $Ax=0$ does not have a solution.

(c.) If $Ax=b$ has a solution, then it is unique

(d.) $y'A=0$ for some nonzero y, where y' denotes the transpose of the vector y.

3. Which of the following matrices has the same row space as of the matrix $\begin{pmatrix}4 & 8 & 4\\ 3 & 6 & 1\\ 2 & 4 & 0\end{pmatrix}$?

(a.) $\begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}$

(b.) $\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

(c.) $\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

(d.) $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$

4. The determinant of the $n\times n$ permutation matrix
$$\begin{bmatrix} & & & & & 1\\ & & & & 1 & \\ & & & \cdot & & \\ & & \cdot & & & \\ & 1 & & & & \\ 1 & & & & & \end{bmatrix}_{n\times n}$$

(a.) $(-1)^n$

(b.) $(-1)^{\left\lceil \frac{n}{2}\right\rceil}$

(c.) -1

(d.) 1

5. The determinant $\begin{vmatrix}1 & 1+x & 1+x+x^2\\ 1 & 1+y & 1+y+y^2\\ 1 & 1+z & 1-+z+z^2\end{vmatrix}$ is equal to

(a.) $(z-y)(z-x)(y-x)$

(b.) $(x-y)(x-z)(y-z)$

(c.) $(x-y)^2(y-z)^2(z-x)^2$

(d.) $(x^2-y^2)(y^2-z^2)(z^2-x^2)$

6. Let A be a 3×4 and b a 3×1 matrix with integer entries. Suppose that the system $Ax=b$ has a complex solution then

(a.) $Ax=b$ has an integer solution

(b.) $Ax=b$ has a rational solution

(c.) The set of real solution to $Ax=0$ has a basis consisting of rational solution

(d.) If $b\neq 0$ then A has positive rank.

7. The matrix $A=\begin{pmatrix}5 & 9 & 8\\ 1 & 8 & 2\\ 9 & 1 & 0\end{pmatrix}$ satisfies:

(a.) A is invertible and the inverse has all integer entries

(b.) Det (A) is odd

(c.) Det(A) is divisible by 13.

(d.) Det (A) has at least two prime divisors.

8. Let A be a 4×7 real matrix and B be a 7×4 real matrix such that $AB=I_4$ where I_4 is the 4×4 identity matrix which of the following is/are always true?

(a.) Rank $(A)=4$

(b.) Rank $(B)=7$

(c.) Nullity $(B)=0$

(d.) $BA=I_7$ where I_7 is the 7×7 identity matrix

9. For the matrix A as given below, which of them satisfy $A^6 = I$?

(a.) $A = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b.) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ 0 & -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$

(c.) $A = \begin{pmatrix} \cos\frac{\pi}{6} & 0 & \sin\frac{\pi}{6} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{6} & 0 & \cos\frac{\pi}{6} \end{pmatrix}$

(d.) $A = \begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} & 0 \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

10. Consider a homogeneous system of linear equation $Ax = 0$ where A is an $m \times n$ real matrix and $n > m$. Then which of the following statements are always true?

(a.) $Ax = 0$ has a solution

(b.) $Ax = 0$ has no nonzero solution

(c.) $Ax = 0$ has a nonzero solution

(d.) Dimension of the space of all solutions is at least $n - m$

11. Let A be a 5×4 matrix with real entries such that $A\underline{x} = \underline{0}$ if and only if $\underline{x} = \underline{0}$ where $\underline{x}$ is a 4×1 vector and $\underline{0}$ is a null vector. Then, the rank of A is

(a.) 4.

(b.) 5.

(c.) 2.

(d.) 1.

12. Let $A = \begin{bmatrix} 1 & 3 & 5 & a & 13 \\ 0 & 1 & 7 & 9 & b \\ 0 & 0 & 1 & 11 & 15 \end{bmatrix}$ where $a, b \in \mathbb{R}$.

Choose the correct statement.

(a.) There exist value of a and b for which the columns of A are linearly independent.

(b.) There exist values of a and b for which $Ax = 0$ has $x = 0$ is the only solution.

(c.) For all values of a and b the rows of A span a 3-dimensional subspace of $\mathbb{R}^5$

(d.) There exist values of a and b for which rank $(A) = 2$.

13. Let S denote the set of all primes p such that the following matrix is invertible when considered as a matrix with entries in $\mathbb{Z}/p\mathbb{Z}$.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ -2 & 0 & 2 \end{pmatrix}$$

Which of the following statements are true?

(a.) S contains all the prime numbers

(b.) S contains all the prime number greater than 10

(c.) S contains all the prime number other than 2 and 5

(d.) S contains all the odd prime numbers.

14. The system of equations

$x + y + z = 1$

$2x + 3y - z = 5$

$x + 2y - kz = 4$ where $k \in \mathbb{R}$

has an infinite number of solutions for

(a.) $k = 0$

(b.) $k = 1$

(c.) $k = 2$

(d.) $k = 3$

15. The determinant of the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ is

(a.) 0

(b.) –9

(c.) –27

(d.) 1

16. Let A be a 5×4 matrix with real entries such that the space of all solutions of the linear system

$AX^t = [1, 2, 3, 4, 5]^t$ is given by $\left\{[1+2s, 2+3s, 3+4s, 4+5s]^t : s \in \mathbb{R}\right\}$. (Here M^t denotes the transpose of a matrix M). Then the rank of A is equal to

(a.) 4

(b.) 3

(c.) 2

(d.) 1

17. Let D be a non zero $n\times n$ real matrix with $n \geq 2$. Which of the following implications is valid?

(a.) $\det(D)=0$ implies ranks $(D)=0$

(b.) $\det(D)=1$ implies ranks $(D)\neq 1$

(c.) $\det(D)=1$ implies ranks $(D)\neq 0$

(d.) $\det(D)=n$ implies ranks $(D)\neq 1$

18. Consider the system of m linear equations in n unknowns given by $Ax=b$, where $A=(a_{ij})$ is a real $m\times n$ matrix, x and b are $n\times 1$ column vectors. Then

(a.) There is at least one solution

(b.) There is at least one solution if b is the zero vector

(c.) If $m=n$ and if the rank of A is n, then there is a unique solution

(d.) If $m<n$ and if the rank of the augmented matrix $[A : b]$ equals the rank of A, then there are infinitely many solutions.

19. The determinant of the following 4×4 matrix $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is

(a.) 4

(b.) 6

(c.) 12

(d.) 24

20. The system of simultaneous linear equations

$$x+y+z=0$$
$$x-y-z=0$$

Has

(a.) No solution in $\mathbf{R}^3$

(b.) A unique solution in $\mathbf{R}^3$

(c.) Infinitely many solutions in $\mathbf{R}^3$

(d.) More than 2 but finitely many solution in $\mathbf{R}^3$

21. Let A and B be upper and lower triangular matrices given by

$$A=\begin{pmatrix} 1 & & & * \\ & 2 & & \\ & & \ddots & \\ 0 & & & n \end{pmatrix} \text{ and}$$

$$B=\begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & \ddots & \\ * & & & n-1 \end{pmatrix}$$

Then

(a.) A is invertible and B is singular

(b.) A is singular and B is invertible

(c.) Both A and B are invertible

(d.) Neither A and B is invertible.

22. A homogenous system of 5 linear equations in 6 variables admits

(a.) No solution in $\mathbf{R}^6$

(b.) A unique solution in $\mathbf{R}^6$

(c.) Infinitely many solution in $\mathbf{R}^6$

(d.) Finite, but more than 2 solutions in $\mathbf{R}^6$

23. Let A be an $m \times n$ a matrix with rank m and B be an $p \times m$ a matrix with rank p. What will be the rank of BA? $(p < m < n)$

(a.) m

(b.) p

(c.) n

(d.) $p+m$

24. Let A be an $n \times n$ matrix and $b = (b_1, b_2, \cdots, b_n)^t$ be a fixed vector. Consider a system of n-linear equation $Ax = b$, where $x = (x_1, x_2, \cdots x_n)^t$. Consider the following statements:

A. If rank $A = n$, the system has a unique solution

B. If rank $A < n$, the system has infinitely many solutions

C. If $b = 0$, the system has at least one solution

Which of the following is correct?

(a.) A and B are true

(b.) A and C are true

(c.) Only A is true

(d.) Only B is true

25. Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} = 3$ for i and j. Then the nullity of A is

(a.) $n-1$

(b.) $n-3$

(c.) n

(d.) 0

26. Let A be a non-zero matrix of order 8 with $A^2 = 0$. Then one of the possible value for rank of A is

(a.) 5

(b.) 6

(c.) 4

(d.) 8

27. The rank of the 4×4 skew symmetric matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix} \text{ is}$$

(a.) 1

(b.) 2

(c.) 3

(d.) 4

28. Let A be a 3×3 matrix and consider the system of equations $Ax = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Then

(a.) If the system is consistent then it has a unique solution

(b.) If A is singular then the system has infinitely many solutions

(c.) If the system is consistent then the $|A| \neq 0$

(d.) If the system has a unique solution then A is non-singular

29. Let S be the solution space of a set of m homogenous linear equations with rea coefficient in n unknowns. If A is the matri of this system equations. Then

(a.) Dimension $S = n -$ rank A

(b.) Dimension S is always n

(c.) Dimension S is infinite

(d.) Dimension $S = n +$ rank A

30. Let A be $n \times n$ matrix over $\mathbb{R}$. Consider th following statements:

1. Rank $A = n$

2. Det $A \neq 0$

Then

(a.) 1 implies 2, but 2 does not imply 1

(b.) 2 implies 1 but 1 does not imply 2

(c.) $1 \Leftrightarrow 2$

(d.) There is no relation between th statements

31. The solutions to the system of equations :

$(1-i)x_1 - ix_2 = 0$

$2x_1 + (1-i)x_2 = 0$ is given by:

(a.) $(x_1, x_2) = (0,\ 0)$

(b.) $(x_1, x_2) = (1,\ 1)$

(c.) $(x_1, x_2) = c\left(1, \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ where c i any complex number.

(d.) $(x_1, x_2) = c\left(\cos\frac{3\pi}{4}, i\sin\frac{3\pi}{4}\right)$ where c i any complex number.

32. Let A be an $m\times n$ matrix where $m<n$. Consider the system of linear equations $A\underset{\sim}{x}=\underset{\sim}{b}$ where b is an $m\times 1$ column vector and $\underset{\sim}{b}\neq 0$. Which of the following is always true?

(a.) The system of equations has no solution.

(b.) The system of equations has a solution if and only if it has infinitely many solutions.

(c.) The system of equations has a unique solution.

(d.) The system of equations has at least one solution.

42. The system $Ax=0$, where A is an $n\times n$ matrix,

(a.) May not have a nonzero solution

(b.) Always has a nonzero solution

(c.) Always has at least 2 linearly independent solutions

(d.) Always has at least n linearly independent solutions

43. Let P be a matrix of order m × n and Q be a matrix of order n × p, n ≠ p. If rank (P) = n and rank (Q) = p, then rank (PQ) is

(a.) n

(b.) p

(c.) np

(d.) n+p

44. Let A be a $m\times n$ matrix with row rank r = column rank. The dimension of the space of solutions of the system of linear equations AX = 0 is

(a.) r

(b.) $n-r$

(c.) $m-r$

(d.) $\min.(m,n)-r$

45. Let M be a $m\times n(m<n)$ matrix with rank m. Then

(a.) For every b in R^m, $Mx=b$ has unique solution

(b.) For every b in R^m, $Mx=b$ has a solution but it is not unique

(c.) There exists $b\in R^m$ for which $Mx=b$ has not solution

(d.) None of the above

46. Consider the system of linear equations

$$a_1x+b_1y+c_1z=d_1,$$

$$a_2x+b_2y+c_2z=d_2,$$

$$a_3x+b_3y+c_3z=d_3,$$

Where a_i,b_i,c_i,d_i are real numbers for $1\leq i\leq 3$. If $\begin{vmatrix} b_1 & c_1 & d_1\\ b_2 & c_2 & d_2\\ b_3 & c_3 & d_3\end{vmatrix}\neq 0$ then the above system has

(a.) At most one solution

(b.) Always exactly one solution

(c.) More than one but finitely many solutions

(d.) Infinitely many solutions.

47. Let $A=\begin{pmatrix} 12 & 24 & 5\\ x & 6 & 2\\ -1 & -2 & 3\end{pmatrix}$. The value of x for which the matrix A is not invertible is

(a.) 6

(b.) 12

(c.) 3

(d.) 2

48. Let $A=\begin{pmatrix} \pi & p\\ q & r\end{pmatrix}$ where p,q,r are rational numbers. If $\det A=0$ and $p\neq 0$ then the value of q^2+r^2

(a.) is 2

(b.) is 1

(c.) is 0

(d.) cannot be determined using the given information

49. Let $A=\begin{bmatrix} a & \pi\\ \pi & \frac{1}{49}\end{bmatrix}$, where a is a real number. Then, A is invertible

(a.) for all $a\neq 22^2$

(b.) for all $a\neq 180^2\times 49$

(c.) for all $a\neq 22^2$ or $a\neq 180^2\times 49$

(d.) for all rational a

50. Let A be an $n \times n$ invertible matrix with integer entries and assume that A^{-1} also only integer entries. Then,

(a.) det $A = n$

(b.) det $A = \pm 1$

(c.) det $A = n^2$

(d.) det A will depend on the entries A and A^{-1}

51. The determinant $\begin{vmatrix} a+b & c+d & e & 1 \\ b+c & d+a & f & 1 \\ c+d & a+b & g & 1 \\ d+a & b+c & h & 1 \end{vmatrix}$ evaluates to

(a.) 0

(b.) 1

(c.) $(a+b)(c+d)+e+f+g+h$

(d.) $(a+b+c+d)(e+f+g+h)$

52. The determinant $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$ evaluates to

(a.) xy

(b.) $(xy)^2$

(c.) $(1-x^2)(1-y^2)$

(d.) x^2+y^2

53. The determinant $\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_0 & x & x_2 & x_3 & x_4 \\ x_0 & x_1 & x & x_3 & x_4 \\ x_0 & x_1 & x_2 & x & x_4 \\ x_0 & x_1 & x_2 & x_3 & x \end{vmatrix}$ evaluates to

(a.) $\left[x_0(x-x_1)(x-x_2)(x-x_3)(x-x_4)\right]^4$

(b.) $x_0(x-x_1)(x-x_2)(x-x_3)(x-x_4)$

(c.) $x_0\left[(x-x_1)(x-x_2)(x-x_3)(x-x_4)\right]^4$

(d.) $xx_0x_1x_2x_3x_4$

54. For real number a, b, c, the following linear system of equations

$x+y+z=1$

$ax+by+cz=1$

$a^2x+b^2y+c^2z=1$

has a unique solution if and only if

(a.) $b=c$ and $b \neq a$.

(b.) $a=b$ and $a \neq c$.

(c.) $a=c$ and $a \neq b$.

(d.) $a \neq b, b \neq c$ and $a \neq c$.

55. If $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ then the rank of (A - I) (I is the 4 × 4 identity matrix) is

(a.) 4

(b.) 3

(c.) 2

(d.) 1

56. Let $M = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Then the rank of M is equal to.

(a.) 3

(b.) 4

(c.) 2

(d.) 1

ASSIGNMENT-4

1. Let S be the set of 3×3 real matrices A with $A^TA=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$. Then the set S contains

(a.) a nilpotent matrix.

(b.) a matrix of rank one.

(c.) a matrix of rank two.

(d.) a non-zero skew-symmetric matrix.

2. An $n\times n$ complex matrix A satisfies $A^k=I_n$, the $n\times n$ identity matrix, where k is a positive integer >1. Suppose 1 is not an eigenvalue of A. Then which of the following statements are necessarily true?

(a.) A is diagonalizable.

(b.) $A+A^2+...+A^{k-1}=0$, The $n\times n$ zero matrixes.

(c.) $tr(A)+tr(A^2)+...+tr(A^{k-1})=-n$.

(d.) $A^{-1}+A^{-2}+...+A^{-(k-1)}=-I_n$.

3. Let $S:\mathbb{R}^n\to\mathbb{R}^n$ be given by $S(v)=\alpha v$ for a fixed $\alpha\in\mathbb{R},\alpha\neq 0$. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a linear transformation such that $B=\{v_1,...,v_n\}$ is a set of linearly independent eigenvectors of T. Then

(a.) The matrix of T with respect to B is diagonal.

(b.) The matrix of $(T-S)$ with respect to B is diagonal.

(c.) The matrix of T with respect to B is not necessarily diagonal, but is upper triangular.

(d.) The matrix of T with respect to B is diagonal, but the matrix of $(T-S)$ with respect to B is not diagonal.

4. Let $A=\begin{pmatrix}a&b&c\\0&a&d\\0&0&a\end{pmatrix}$ be a 3×3 matrix where a,b,c,d are integers. Then, we must have:

(a.) If $a\neq 0$, there is a polynomial $p\in\mathbb{Q}[x]$ such that $p(A)$ is the inverse of A.

(b.) For each polynomial $q\in\mathbb{Z}[x]$, the matrix $q(A)=\begin{pmatrix}q(a)&q(b)&q(c)\\0&q(a)&q(d)\\0&0&q(a)\end{pmatrix}$.

(c.) If $A^n=0$ for some positive integer n, then $A^3=0$.

(d.) A commutes with every matrix of the form $\begin{pmatrix}a'&0&c'\\0&a'&0\\0&0&a'\end{pmatrix}$.

5. Let A be an invertible 4×4 real matrix. Which of the following are NOT true?

(a.) Rank $A=4$.

(b.) For every vector $b\in\mathbb{R}^4$, $Ax=b$ has exactly one solution.

(c.) $\dim(\text{nullspace } A)\geq 1$.

(d.) 0 is an eigenvalue of A.

6. Let $\underline{u}$ be a real $n\times 1$ vector satisfying $\underline{u}'\underline{u}=1$, where $\underline{u}'$ the transpose of is $\underline{u}$. Define $A=I-2\underline{u}\,\underline{u}'$ where I is the n^{th} order identity matrix. Which of the following statements are true?

(a.) A is singular.

(b.) $A^2=A$.

(c.) Trace $(A)=n-2$.

(d.) $A^2=I$.

7. Let P be a 2×2 complex matrix such that p^*p is the identity matrix, where p^* is the conjugate transpose of P. Then the Eigen values of P are

(a.) real

(b.) complex conjugates of each other

(c.) reciprocals of each other

(d.) of modulus 1

8. Which of the following are eigenvalues of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(a.) +1

(b.) −1

(c.) $+i$

(d.) $-i$

9. Let J denote a 101×101 matrix with all the entries equal to 1 and let I denote the identity matrix of order 101. Then the determinant of $J-I$ is

(a.) 101

(b.) 1

(c.) 0

(d.) 100

10. Let A be a 5×5 matrix with real entries such that the sum of the entries in each row of A is 1. Then the sum of all the entries in A^3 is

(a.) 3

(b.) 15

(c.) 5

(d.) 125

11. For a fixed positive integer $n\geq 3$, let A be the $n\times n$ matrix defined as $A=I-\frac{1}{n}J$, where I is the identity matrix and J is the $n\times n$ matrix with all entries equal to 1. Which of the following statements is NOT true?

(a.) $A^k=A$ for every positive integer k.

(b.) Trace $(A)=n-1$

(c.) Rank $(A)+$ Rank $(I-A)=n$.

(d.) A is invertible.

12. Let A be an $n\times n$ matrix with real entries. Which of the following is correct?

(a.) If $A^2=0$, then A is diagonalizable over complex numbers.

(b.) If $A^2=I$, then A is diagonalizable over real numbers.

(c.) If $A^2=A$, then A is diagonalizable only over complex numbers.

(d.) The only matrix of size n satisfying the characteristic polynomial of A is A.

13. Let A be a 4×4 invertible real matrix. Which of the following is NOT necessarily true?

(a.) The rows of A form a basis of $\mathbb{R}^4$.

(b.) Null space of A contains only the 0 vector.

(c.) A has 4 distinct eigenvalues.

(d.) Image of the linear transformation $x\mapsto Ax$ on $\mathbb{R}^4$ is $\mathbb{R}^4$.

14. Let $A\in M_{10}(\mathbb{C})$, the vector space of 10×10 matrices with entries in $\mathbb{C}$. Let W_A be the subspace of $M_{10}(\mathbb{C})$ spanned by $\{A^n \mid n\geq 0\}$. Choose the correct statements

(a.) For any A, $\dim(W_A)\leq 10$

(b.) For any A, $\dim(W_A)<10$

(c.) For some A, $10<\dim(W_A)<100$

(d.) For some A, $\dim(W_A)=100$

15. Let A be a complex 3×3 matrix with $A^3=-I$. Which of the following statements are correct?

(a.) A has three distinct eigenvalues

(b.) A is diagonalizable over $\mathbb{C}$

(c.) A is triangularizable over $\mathbb{C}$

(d.) A is non-singular

16. Let $f(x)$ be the minimal polynomial of the 4×4 matrix $A=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$. Then the rank of 4×4 matrix $f(A)$ is

(a.) 0

(b.) 1

(c.) 2

(d.) 4

17. Let n be an integer ≥ 2 and let $M_n(\mathbb{R})$ denote the vector space of $n\times n$ real matrices. Let $B\in M_n(\mathbb{R})$ be an orthogonal matrix and let B^t denote the transpose of B. Consider $W_B=\{B^tAB : A\in M_n(\mathbb{R})\}$. Which of the following are necessarily true?

(a.) W_B is a subspace of $M_n(\mathbb{R})$ and dim $W_B \leq$ rank (B)

(b.) W_B is a subspace of $M_n(\mathbb{R})$ and dim $W_B =$ rank (B) rank (B^t)

(c.) $W_B = M_n(\mathbb{R})$

(d.) W_B is not a subspace of $M_n(\mathbb{R})$.

18. Let A be a non-zero linear transformation on a real vector space V of dimension n. Let the subspace $V_\circ \subset V$ be the image of V under A. Let $k=$ dim $V_\circ < n$ and suppose that for some $\lambda \subset R$. $A^2=\lambda A$. Then

(a.) $\lambda = 1$

(b.) Det $A=|\lambda|^n$

(c.) λ is the only eigenvalue of A.

(d.) There is a nontrivial subspace $V_t \subset V$ such that Ax=0 for all $x\in V_1$

19. Let C be a $n\times n$ real matrix. Let W be the vector space spanned by $\{I, C, C^2 ......C^{2n}\}$ the dimension of the vector space W is

(a.) $2n$

(b.) At most n

(c.) n^2

(d.) At most $2n$

20. Let N be a nonzero 3×3 matrix with the property $N^2=0$ .which of the following is/are true?

(a.) N is not similar to a diagonal matrix.

(b.) N is similar to a diagonal matrix.

(c.) N has one non-zero eigenvector.

(d.) N has three linearly independent eigenvector.

21. Suppose the matrix $A=\begin{pmatrix}40&-29&-11\\-18&30&-12\\26&24&-50\end{pmatrix}$ has a certain complex number $\lambda\neq0$ as an eigenvalue. With of the following numbers must also be an eigenvalue of A?

(a.) $\lambda+20$

(b.) $\lambda-20$

(c.) $20-\lambda$

(d.) $-20-\lambda$

22. Let A be a 3×3 matrix with real entries such that $\det(A)=6$ and the trace of A is 0. If $\det(A+I)=0$, where I denotes the 3×3 identity matrix, then the eigenvalues of A are

(a.) $-1, 2, 3$

(b.) $-1, 2, -3$

(c.) $1, 2, -3$

(d.) $-1, -2, 3$

23. Consider a matrix $A=(a_{ij})_{n\times n}$ with integer entries such that

$a_{ij}=0$ for $i>j$ and $a_{ii}=1$ for $i=1,....,n$.

Which of the following properties must be true?

(a.) A^{-1} exists and it has integer entries

(b.) A^{-1} exists and it has some entries that are not integers

(c.) A^{-1} is a polynomial function of A with integer coefficients

(d.) A^{-1} is not a power of A unless A is the identity matrix

24. Let ω be a complex number such that $\omega^3 = 1$, but $\omega \neq 1$. If

$$A = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{bmatrix},$$

Then which of the following statements are true?

(a.) A is invertible

(b.) $\operatorname{rank}(A) = 2$

(c.) 0 is an eigenvalues of A

(d.) There exist linearly independent vectors $v, w \in \mathbb{C}^3$ such that $Av = Aw = 0$

25. Let A be a 4×4 matrix with real entries such that $-1, 1, 2, -2$ are its eigenvalues. If $B = A^4 - 5A^2 + 5I$, where I denotes the 4×4 identity matrix, then which of the following statements are correct?

(a.) $\det(A + B) = 0$

(b.) $\det(B) = 1$

(c.) trace of $A - B$ is 0

(d.) trace of $A + B$ is 4

26. Let $M_2(\mathbb{R})$ denote the set of 2×2 real matrices. Let $A \in M_2(\mathbb{R})$ be of trace 2 and determinant -3. Identifying $M_2(\mathbb{R})$ with $\mathbb{R}^4$, consider the linear transformation $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(B) = AB$. Then which of the following statements are true?

(a.) T is diagonalizable

(b.) 2 is an eigenvalues of T

(c.) T is invertible

(d.) $T(B) = B$ for some $0 \neq B$ in $M_2(\mathbb{R})$

27. Let A be a 2×2 non-zero matrix with entries in $\mathbb{C}$ such that $A^2 = 0$. Which of the following statements must be true?

(a.) PAP^{-1} is diagonal for some invertible 2×2 matrix P with entries in $\mathbb{R}$

(b.) A has two distinct eigenvalues in $\mathbb{C}$

(c.) A has only one eigenvalue in $\mathbb{C}$ with multiplicity 2

(d.) $Av = v$ for some $v \in \mathbb{C}^2, v \neq 0$

28. Let N be a 3 × 3 non zero matrix with the property $N^3 = 0$. Which of the following is/are true?

(a.) N is not similar to a diagonal matrix

(b.) N is similar to a diagonal matrix

(c.) N has one non-zero eigenvector

(d.) N has three linearly independent eigenvector.

29. Suppose A is an $n \times n$ real symmetric matrix with eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ then

(a.) $\prod_{i=1}^{n} \lambda_1 < \det(A)$

(b.) $\prod_{i=1}^{n} \lambda_1 > \det(A)$

(c.) $\prod_{i=1}^{n} \lambda_1 = \det(A)$

(d.) If $\det(A) = 1$ then $\lambda_j = 1$ for $j = 1, \ldots, n$

30. Let A be an $n \times n$ matrix, $n \geq 2$, with characteristic polynomial $x^{n-2}(x^2 - 1)$. Then

(a.) $A^n = A^{n-2}$

(b.) Rank of A is 2

(c.) Rank of A is at least 2

(d.) There exist non-zero vectors x and y such that $A(x + y) = x - y$

31. Let V be the vector space of polynomials of degree ≤ 5 over $\mathbf{R}$. Let $D : V \to V$ be the derivative map $p \to p'$. Then

(a.) 0 is the only eigenvalue of D

(b.) 1 is an eigenvalue of D

(c.) 5 is an eigenvalue of D

(d.) D is invertible

32. If A is a real 2×2 matrix such that $A^2 - A = 0$, then

(a.) Either $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(b.) There are infinitely many such matrices A

(c.) There are only finitely many such matrices A

(d.) A has to be a diagonal matrix.

33. Given that a 3×3 matrix satisfies the equation $A^3-A^2+A-I=0$, the value of A^4 is

(a.) Not computable from the given data

(b.) $-A^3+A^2-A-I$

(c.) 0

(d.) I

34. Let A be the $n\times n$ matrix with all entries equal to 1. The eigen values of A are

(a.) 0 with multiplicity $(n-1)$ and n with multiplicity 1

(b.) 0 with multiplicity 1 and n with multiplicity $(n-1)$

(c.) 0 with multiplicity 1 and 1 with multiplicity 1

(d.) n with multiplicity 1 and 1 with multiplicity 1.

35. A square matrix A is said to be idempotent if $A^2=A$. An idempotent matrix is non-singular if and only if

(a.) All eigen values are real

(b.) All eigen values are non-negative

(c.) All eigen values are either 1 or 0

(d.) All eigen values are 1.

36. Let A be a symmetric idempotent matrix. Which of the following is not true?

(a.) $\det(A)\neq0\Rightarrow A=I$

(b.) Trace A = Rank A

(c.) If the i^{th} diagonal element of A is zero, at least one element in the i^{th} row is non-zero.

(d.) Every latent/characteristic root of A is either 0 or 1.

37. Let $A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$. Then the eigen values of A are

(a.) 2, 1 and 0

(b.) 2, 1 + i and 1-i

(c.) 2, -1 and -1

(d.) 1, -1 and 0

38. Let $A=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$ be such that A has real eigenvalues. Then

(a.) $\theta=n\pi$ for some integer n

(b.) $\theta=2n\pi+\frac{\pi}{2}$ for some integer n

(c.) There is no restriction on θ

(d.) $\theta=2n\pi+\frac{\pi}{2}$ for some integer n

39. Let $A=(a_{ij})$ be an $n\times n$ matrix with real entries such that the sum of all the entries in each row is zero. Consider the following statements:

A. A is non-singular

B. A is singular

C. 0 is an eigen value of A

Which of the following is correct?

(a.) Only A is true

(b.) A and C are true

(c.) B and C are true

(d.) Only C is true.

40. Let $A=\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}$. Then the minimal polynomial of A is

(a.) $\lambda^2-4\lambda-5$

(b.) $\lambda^2-5\lambda-4$

(c.) $\lambda^2-3\lambda^2-9\lambda-5$

(d.) $\lambda^3+3\lambda^2-9\lambda+5$

41. Let $A=(a_{ij})$ be an $n\times n$ matrix such that $a_{ij}=3$ for i and j. Then the nullity of A is

(a.) $n-1$

(b.) $n-3$

(c.) n

(d.) 0

42. Let A be a non-zero matrix of order 8 with $A^2=0$. Then one of the possible value for rank of A is

(a.) 5

(b.) 6

(c.) 4

(d.) 8

43. A basis of the eigenspace of the matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ consist of

(a.) $\begin{vmatrix}1\\1\end{vmatrix}$

(b.) $\begin{vmatrix}0\\0\end{vmatrix}$ and $\begin{vmatrix}1\\1\end{vmatrix}$

(c.) $\begin{vmatrix}0\\1\end{vmatrix}$

(d.) $\begin{vmatrix}0\\1\end{vmatrix}$ and $\begin{vmatrix}1\\0\end{vmatrix}$

44. If $\begin{vmatrix}1\\-1\end{vmatrix}$ is an eigenvector of $A=\begin{bmatrix}1 & -1\\ -3 & 2n\end{bmatrix}$ then n is

(a.) 1

(b.) 2

(c.) 3

(d.) None of these

45. The characteristic roots of the 3×3 matrix A are 3, 2 and -1. If $B=A^2-A$ then the determinant of B is

(a.) 24

(b.) -2

(c.) 12

(d.) -15

46. Let A be a real 4×4 matrix with characteristic polynomial $P(T)=\left(T^2+1\right)^2$. Which of the following is true?

(a.) A is diagonalizable over complex numbers but not over real numbers

(b.) A is nilpotent

(c.) A is invertible

(d.) There is no such matrix A

47. If the characteristic roots of $\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ are λ_1 and λ_2, the characteristic roots of $\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$ are

(a.) $\lambda_1,\lambda_2,\lambda_1-\lambda_2$

(b.) λ_1 and λ_2

(c.) $2\lambda_1$ and $2\lambda_2$

(d.) $\lambda_1+\lambda_2$ and $|\lambda_1-\lambda_2|$

48. If $A=\begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ which of the following is the zero matrix?

(a.) A^2+A-5I

(b.) A^2+A-5I

(c.) A^2+A-I

(d.) $A^2+3A-5I$

49. If $T:V\to V$ is a line operator for $Dim\,V=n$ and T has a distinct eigenvalues, then

(a.) T must be invertible

(b.) T must be diagonalizable

(c.) T must be invertible as well as diagonalizable

(d.) T is not diagonalizable

50. The matrix $\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & 0\\ 0 & 0 & 4\end{bmatrix}$ is similar to the matrix

(a.) $\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{bmatrix}$

(b.) $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 3\end{bmatrix}$

(c.) $\begin{bmatrix}0 & 1 & 0\\ -1 & 4 & 0\\ 0 & 0 & 0\end{bmatrix}$

(d.) $\begin{bmatrix}3 & 0 & 0\\ 0 & 1 & 2\\ 0 & -4 & 4\end{bmatrix}$

51. Let V be the vector space $\mathbb{R}^2$ and let T be the linear transformation on V defined by $T(x,y)=(x+y,y)$. Then the characteristic polynomial of T is

(a.) $1+3x+x^2$

(b.) $2+2x$

(c.) $1+2x+x^2$

(d.) $1+x^2$

52. A 2×2 real matrix A is diagonalizable then :

(a.) $(tr\ A)^2 \le 4$ Det A

(b.) $(tr\ A)^2 \ge 4$ Det A

(c.) $(trA)^2 \le 4Det\ A$

(d.) $Tr\ A = Det\ A$

53. Let A be a 3×3 complex matrix such that $A^3=I$ (= the 3×3 identity matrix). Then :

(a.) A is diagnonalizable.

(b.) A is not diagonalizable.

(c.) The minimal polynomial of A has a repeated root.

(d.) All eigenvalues of A are real.

54. The minimal polynomial of the 3×3 real matrix $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & a & b \end{pmatrix}$ is

(a.) $(X-a)(X-b)$

(b.) $(X-a)^2(X-b)$

(c.) $(X-a)^2(X-b)^2$

(d.) $(X-a)(X-b)^2$

55. The characteristic polynomial of the 3×3 real matrix $A=\begin{pmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{pmatrix}$ is

(a.) X^3+aX^2+bX+c

(b.) $(X-a)(X-b)(X-c)$

(c.) $(X-1)(X-abc)^2$

(d.) $(X-1)^2(X-abc)$

56. Let M be a 3×3 matrix and suppose that $1,2$ and 3 are the eigenvalues of M. If

$$M^{-1}=\frac{M^2}{\alpha}-M+\frac{11}{\alpha}I_3$$

for some scalar $\alpha \ne 0$, then α is equal to

____

57. Let M be a 3×3 singular matrix and suppose that 2 and 3 are eigenvalues of M. Then the number of linearly independent eigenvectors of M^3+2M+I_3 is equal to

____

58. Let M be a 3×3 matrix such that $M\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}$ and suppose that $M^3\begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix}=\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ for some $\alpha,\beta,\gamma \in \mathbb{R}$. Then $|\alpha|$ is equal to __________

59. Let $T:\mathbb{R}^4 \to \mathbb{R}^4$ be a linear map such that the null space of T is $\{(x,y,z,w)\in\mathbb{R}^4 : x+y+z+w=0\}$ and the rank of $(T-4I_4)$ is 3. If the minimal polynomial of T is $x(x-4)^{\alpha}$, then α is equal to __________

60. Let M be an invertible Hermitian matrix and let $x,y\in\mathbb{R}$ be such that $x^2<4y$. Then

(a.) both $M^2+xM+yI$ and $M^2-xM+yI$ are singular

(b.) $M^2+xM+yI$ is singular but $M^2-xM+yI$ is non-singular

(c.) $M^2+xM+yI$ is non-singular but $M^2-xM+yI$ is singular

(d.) both $M^2+xM+yI$ and $M^2-xM+yI$ are non-singular

61. Let $A\in M_3(\mathbb{R})$ be such that det $(A-I)=0$ where I denotes the 3×3 identity matrix. If the trace$(A)=13$ and det $(A)=32$, then the sum of squares of the eigenvalues of A is__________

62. The possible set of eigen values of a 4×4 skew-symmetric orthogonal real matrix is

(a.) $\{\pm i\}$

(b.) $\{\pm i, \pm 1\}$

(c.) $\{\pm 1\}$

(d.) $\{0, \pm i\}$

63. Let V be a vector space of dimension $m \geq 2$. Let $T : V \to V$ be a linear transformation such that $T^{n+1} = 0$ and $T^n \neq 0$ for some $n \geq 1$. Then which of the following is necessarily **TRUE**?

(a.) Rank $(T^n) \leq$ Nullity (T^n)

(b.) Trace $(T) \neq 0$

(c.) T is diagonalizable

(d.) $n = m$

64. For the matrix $M = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$,

Which of the following statements are correct?

P: M is skew-Hermitian and iM is Hermitian

Q: M is Hermitian and iM is skew Hermitian

R: eigenvalues of M are real

S: eigenvalues of iM are real

(a.) P and R only

(b.) Q and R only

(c.) P and S only

(d.) Q and S only

65. The distinct eigenvalues of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are

(a.) 0 and 1

(b.) 1 and -1

(c.) 1 and 2

(d.) 0 and 2

66. The minimal polynomial associated with the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is

(a.) $x^3 - x^2 - 2x - 3$

(b.) $x^3 - x^2 + 2x - 3$

(c.) $x^3 - x^2 - 3x - 3$

(d.) $x^3 - x^2 + 3x - 3$

67. Which of the following matrices is NOT diagonalizable?

(a.) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

(b.) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$

(c.) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(d.) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

68. The number of linearly independent eigenvectors of the matrix $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ is

(a.) 1

(b.) 2

(c.) 3

(d.) 4

69. Consider the matrix $N = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then N is

(a.) Non-invertible

(b.) Skew-symmetric

(c.) Symmetric

(d.) Orthogonal

70. If M is any 3×3 real matrix and $N=\begin{bmatrix}3/5 & -4/5 & 0\\ 4/5 & 3/5 & 0\\ 0 & 0 & 1\end{bmatrix}$, then trace $\left(NMN^T\right)$ is equal to

(a.) $\left[trace(N)\right]^2 trace(M)$

(b.) $2trace(N)+trace(M)$

(c.) $trace(M)$

(d.) $\left[trace(N)\right]^2+trace(M)$

71. Let A be a 3×3 matrix. Suppose that the eigen values of A are $-1,0,1$ with respective eigen vectors $(1,-1,0)^t$, $(1,1-2)^t$ and $(1,1,1)^t$. Then $6A$ equals

(a.) $\begin{bmatrix}-1 & 5 & 2\\ 5 & -1 & 2\\ 2 & 2 & 2\end{bmatrix}$

(b.) $\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$

(c.) $\begin{bmatrix}1 & 5 & 3\\ 5 & 1 & 3\\ 3 & 3 & 3\end{bmatrix}$

(d.) $\begin{bmatrix}-3 & 9 & 0\\ 9 & -3 & 0\\ 0 & 0 & 6\end{bmatrix}$

72. Let $M=\begin{pmatrix}1 & a & b\\ 0 & 2 & c\\ 0 & 0 & 1\end{pmatrix}$, $a,b,c\in\mathbb{R}$ .Then, M is diagonalizable, if and only if

(a.) $a=bc$

(b.) $b=ac$

(c.) $c=ab$

(d.) $a=b=c$

73. Let M be the real 5×5 matrix having all of its entries equal to 1. Then,

(a.) M is not diagonalizable

(b.) M is idempotent

(c.) M is nilpotent

(d.) The minimal polynomial and the characteristic polynomial of M are not equal

74. Let $M=\begin{pmatrix}1 & 3 & 3\\ 0 & 4 & 5\\ 0 & 0 & 9\end{pmatrix}$. Then

(a.) M is diagonalizable but not M^2

(b.) M^2 is diagonalizable but not M

(c.) Both M and M^2 are diagonalizable

(d.) Neither M nor M^2 is diagonalizable

75. Let M be a skew symmetric, orthogonal real matrix, The only possible eigen values are

(a.) $-1,1$

(b.) $-i,i$

(c.) 0

(d.) $1,i$

76. If a square matrix of order 10 has exactly 4 distinct eigen values, then the degree of its minimal polynomial is

(a.) Least 4

(b.) At most 4

(c.) At least 6

(d.) At most 6

77. Let P be a $n\times n$ matrix with integral entries and $Q=P+\frac{1}{2}I$, where I denotes the $n\times n$ identity matrix. Then Q is

(a.) Idempotent, i.e $Q^2=Q$

(b.) Invertible

(c.) Nilpotent

(d.) Unipotent, i.e., $Q-I$ is nilpotent

78. Let M be a square matrix of order, 2 such that rank of M is 1. Then M is

(a.) Diagonalizable and nonsingular

(b.) Diagonalizable and nilpotent

(c.) Neither diagonalizable nor nilpotent

(d.) Either diagonalizable or nilpotent but not both

79. Let the characteristics equation of a matrix M be $\lambda^2-\lambda-1=0$, then

(a.) M^{-1} does not exist

(b.) M^{-1} exists but cannot be determined from the data

(c.) $M^{-1}=M+1$

(d.) $M^{-1}=M-1$

80. Consider the matrix $M=\begin{pmatrix}1&0&-1\\0&1&0\\1&1&-1\end{pmatrix}$ and let S_M be the set of 3×3 matrices N such that $M.N=0$. Then the dimension of the real vector space S_M is equal to

(a.) 0

(b.) 1

(c.) 2

(d.) 3

81. Consider the matrix $M=\begin{pmatrix}a^2&ab&ac\\ab&b^2&bc\\ac&bc&c^2\end{pmatrix}$ where, $a,b\,\&\,c$ are non-zero real numbers.

Then the matrix has

(a.) Three non-zero real eigen values

(b.) Complex eigen values

(c.) Two non-zero eigen value

(d.) Only one non-zero eigen value

82. Let A be a non-zero upper triangular matrix all of whose eigen values are 0. Then $I+A$ is

(a.) Invertible

(b.) Singular

(c.) Idempotent

(d.) Nilpotent

83. The eigen values of a skew-symmetric matrix are

(a.) Negative

(b.) Real

(c.) Of a absolute value 1

(d.) Purely imaginary or zero

84. Let A be a 3×3 matrix with eigen values 1, –1, 0. Then the determinant of $I+A^{100}$ is

(a.) 6

(b.) 4

(c.) 9

(d.) 100

85. The eigen values of a 3×3 real matrix P are Then

(a.) $P^{-1}=\frac{1}{6}\left(5I+2P-P^2\right)$

(b.) $P^{-1}=\frac{1}{6}\left(5I-2P+P^2\right)$

(c.) $P^{-1}=\frac{1}{6}\left(5I-2P-P^2\right)$

(d.) $P^{-1}=\frac{1}{6}\left(5I+2P+P^2\right)$

86. Let $T:\mathbb{C}^n\to\mathbb{C}^n$ be a linear operator having n distinct eigen values. Then

(a.) T is invertible

(b.) T is invertible as well as diagonalizable

(c.) T is not diagonalizable

(d.) T is diagonalizable

87. Let U be a 3×3 complex Hermitian matrix which is Unitary. Then the distinct eigen values of U are

(a.) $\pm i$

(b.) $1\pm i$

(c.) ±1

(d.) $\frac{1}{2}(1\pm i)$

88. Let A be an $n\times n$ complex matrix whose characteristic polynomial is given by $f(t)=t^n+c_{n-1}t^{n-1}+\ldots\ldots+c_1t+c_0$.

Then

(a.) $\det(A)=c_{n-1}$

(b.) $\det(A)=c_0$

(c.) $\det(A)=(-1)^n c_{n-1}$

(d.) $\det(A)=(-1)^n c_0$

89. Let $T:C^n\to C^n$ be a linear operator rank $n-2$. Then

(a.) 0 is not an eigen value of T

(b.) 0 must be an eigen value of T

(c.) 1 can never be an eigen value of T

(d.) 1 must be an eigen value of T

90. For $0<\theta<\pi$, the matrix $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

(a.) Has no real eigen value
(b.) Is orthogonal
(c.) Is symmetric
(d.) Is skew symmetric

91. Let P and Q be square matrices such that $PQ=I$, the identity matrix. Then zero is an eigen value of

(a.) P but not of Q
(b.) Q but not of P
(c.) Both P and Q
(d.) Neither P nor Q

92. Let A and B be $n\times n$ matrices with the same minimal polynomial. Then

(a.) A is similar to B
(b.) A is diagonalizable iff B is diagonalizable
(c.) $A-B$ is singular
(d.) A and B commute

93. Let A be an n ×n matrix which is both Hermitian and unitary. Then

(a.) $A^2=I$
(b.) A is real
(c.) The eigen value of A are 0,1,–1
(d.) The characteristic and minimal polynomials of A are the same.

94. A matrix M has eigen values 1 and 4 with corresponding eigen vectors $(1,-1)^T$ and $(2,1)^T$, respectively. Then M is

(a.) $\begin{pmatrix}-4 & -8\\ 5 & 9\end{pmatrix}$
(b.) $\begin{pmatrix}9 & -8\\ 5 & -4\end{pmatrix}$
(c.) $\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}$
(d.) $\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}$

ASSINGMENT-5

1. Consider the quadratic form $Q(v) = v^t A v$, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad v = (x, y, z, w)$$

Then

(a) Q has rank 3.

(b) $xy + z^2 = Q(Pv)$ for some invertible 4×4 real matrix P.

(c) $xy + y^2 + z^2 = Q(Pv)$ for some invertible 4×4 real matrix P.

(d) $x^2 + y^2 - zw = Q(Pv)$ for some invertible 4×4 real matrix P.

2. Let u, v, w be vectors in an inner-product space V, satisfying $\|u\| = \|v\| = \|w\| = 2$ and $<u, v> = 0$, $<u, w> = 1$, $<v, w> = -1$. Then which of the following are true?

(a) $\|w + v - u\| = 2\sqrt{2}$

(b) $\left\{\frac{1}{2}u, \frac{1}{2}v\right\}$ forms an orthonormal basis of a two dimensional subspace of V

(c) w and $4u - w$ are orthogonal to each other.

(d) u, v, w are necessarily linearly independent.

3. Let V denote the vector space of all polynomials over R of degree less than or equal to n. Which of the following defines a norm on V?

(a) $\|p\|^2 = |p(1)|^2 + \ldots + |p(n+1)|^2, p \in V$

(b) $\|p\| = \sup_{t \in [0,1]} |p(t)|, \ p \in V$

(c) $\|p\| = \int_0^1 |p(t)| dt, \ p \in V$

(d) $\|p\| = \sup_{t \in [0,1]} |p'(t)|, \ p \in V$

4. For any real square matrix M let $\lambda^+(M)$ b the number of positive eigenvalues of M counting multiplicities. Let A be an $n \times n$ real symmetric matrix and Q be an $n \times n$ real invertible matrix. Then

(a.) Rank A = Rank $Q^T AQ$.

(b.) Rank A = Rank $Q^{-1} AQ$.

(c.) $\lambda^+(A) = \lambda^+(Q^T AQ)$.

(d.) $\lambda^+(A) = \lambda^+(Q^{-1} AQ)$.

5. Let y be a nonzero vector in an inne product space V. Then which of th following are subspaces of V?

(a.) $\{x \in V \mid \langle x, y \rangle = 0\}$.

(b.) $\{x \in V \mid \langle x, y \rangle = 1\}$.

(c.) $\{x \in V \mid \langle x, y \rangle = 0$ for all z such that $\langle z, y \rangle = 0\}$.

(d.) $\{x \in V \mid \langle x, z \rangle = 1$ for all z such that $\langle z, y \rangle = 1\}$.

6. Consider $\mathbb{R}^3$ with the standard inner product. Let W be the subspace of $\mathbb{R}^3$ spanned by $(1, 0, -1)$. Which of the following is a basis for the orthogonal complement of W?

(a) $\{(1, 0, 1), (0, 1, 0)\}$

(b) $\{(1, 2, 1), (0, 1, 1)\}$

(c) $\{(2, 1, 2), (4, 2, 4)\}$

(d) $\{(2, -1, 2), (1, 3, 1), (-1, -1, -1)\}$

7. Consider the quadratic forms q and p given by $q(x, y, z, w) = x^2 + y^2 + z^2 + bw^2$ and $p(x, y, z, w) = x^2 + y^2 + cwz$.

Which of the following statements are true?

(a) p and q are equivalent over $\mathbb{C}$ if b and c are non-zero complex numbers.

(b) p and q are equivalent over $\mathbb{R}$ if b and c are non-zero real numbers.

(c) p and q are equivalent over $\mathbb{R}$ if b and c are non-zero real numbers with b negative.

(d) p and q are NOT equivalent over $\mathbb{R}$ if $c = 0$

8. Let V be the inner product space consisting of linear polynomials, $p:[0,\ 1]\to\mathbb{R}$ (i.e., V consists of polynomials p of the form $p(x)=ax+b,\ a,b\in\mathbb{R}$), with the inner product defined by $\langle p,\ q\rangle=\int_0^1 p(x)q(x)dx$ for $p,q\in V$. An orthonormal basis of V is

(a) $\{1,\ x\}$

(b) $\{1,\ x\sqrt{3}\}$

(c) $\{1,\ (2x-1)\sqrt{3}\}$

(d) $\left\{1,\ x-\frac{1}{2}\right\}$

9. Let a,b,c be positive real numbers such that $b^2+c^2<a<1$. Consider the 3×3 matrix

$$A=\begin{bmatrix}1 & b & c\\ b & a & 0\\ c & 0 & 1\end{bmatrix}.$$

(a) All the eigenvalues of A are negative real numbers

(b) All the eigenvalues of A are positive real numbers

(c) A can have a positive as well as a negative eigenvalue

(d) Eigenvalues of A can be non-real complex numbers.

10. Let $A=(a_{ij})$ be an $n\times n$ complex matrix and let A^* denote the conjugate transpose of A. Which of the following statements are necessarily true?

(a) If A is invertible, then $\operatorname{tr}(A^*A)\neq 0$, i.e., the trace of A^*A is non zero.

(b) If $\operatorname{tr}(A^*A)\neq 0$, then A is invertible.

(c) If $|\operatorname{tr}(A^*A)|<n^2$, then $|a_{ij}|<1$ for some i,j

(d) If $\operatorname{tr}(A^*A)=0$, then A is the zero matrix.

11. Suppose A is a 3×3 symmetric matrix such that $[x,y,\ 1]A\begin{bmatrix}x\\ y\\ 1\end{bmatrix}=xy-1$. Let p be the number of positive eigenvalues of A and let $q=$ rank $(A)-p$. Then

(a) $p=1$

(b) $p=2$

(c) $q=2$

(d) $q=1$.

12. Let ξ be a primitive cube root of unity. Define $A=\begin{pmatrix}\xi^{-1} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \xi\end{pmatrix}$. For a vector $v=(v_1,v_2,v_3)\in\mathbb{R}^3$ define $|v|_A=\sqrt{|vAv^T|}$ where v^T is transpose of v. If $w=(1,1,1)$ then $|w|_A$ equals

(a) 0

(b) 1

(c) –1

(d) 2

13. Which of the following matrices are positive definite?

(a) $\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$

(b) $\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$

(c) $\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}$

(d) $\begin{bmatrix}0 & 4\\ 4 & 0\end{bmatrix}$

14. Consider the quadratic form $q(x,y,z)=4x^2+y^2-z^2+4xy-2xz-yz$ over $\mathbb{R}$. Which of the following statements about the range of values taken by q as x,y,z very over $\mathbb{R}$, are true?

(a) Range contains $[1,\ \infty)$

(b) Range is contained in $[0,\ \infty)$

(c) Range $=\mathbb{R}$

(d) Range is contained in $[-N,\ \infty)$ for some large natural number N depending on q

15. Let J be the 3×3 matrix all of whose entries are 1. Then:

(a) 0 and 3 are the only eigenvalues of A

(b) J is positive semi-definite, i.e., $\langle Jx, x\rangle \geq 0$ for all $x \in \mathbb{R}^3$

(c) J is diagonalizable

(d) J is positive definite, i.e., $\langle Jx, x\rangle > 0$ for all $x \in \mathbb{R}^3$ with $x \neq 0$

16. Let ζ be a primitive fifth root of unity. Define

$$A = \begin{pmatrix} \zeta^{-2} & 0 & 0 & 0 & 0 \\ 0 & \zeta^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & \zeta^2 \end{pmatrix}$$

For a vector $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbf{R}^5$, define $|v|_A = \sqrt{|vAv^T|}$ where v^T is transpose of v.

If $w = (1, -1, 1, 1, -1)$, then $|w|_A$ equals

(a) 0

(b) 1

(c) –1

(d) 2

17. Let $a_{ij} = a_i a_j$, $1 \leq i, j \leq b$, where $a_1, \ldots, a_n$ are real numbers. Let $A = ((a_{ij}))$ be the $n \times n$ matrix $((a_{ij}))$. Then

(a) It is possible to choose $a_1, \ldots, a_n$ so as to make the matrix A non singular

(b) The matrix A is positive definite if $(a_1, \ldots, a_n)$ is a non zero vector

(c) The matrix A is positive semi definite for all $(a_1, \ldots, a_n)$

(d) For all $(a_1, \ldots, a_n)$, zero is an eigenvalue of A.

18. Suppose A, B are $n \times n$ positive definite matrices and I be the $n \times n$ identity matrix. Then which of the following are positive definite.

(a) $A + B$

(b) ABA^*

(c) $A^2 + I$

(d) AB

19. Let T be a linear transformation on the re vector space $\mathbf{R}^n$ over $\mathbf{R}$ such that $T^2 = \lambda$ for same $\lambda \in \mathbf{R}$. Then

(a) $\|Tx\| = |\lambda| \|x\|$ for all $x \in \mathbf{R}^n$

(b) If $\|Tx\| = \|x\|$ for some non zero vect $x \in \mathbf{R}^n$, then $\lambda = \pm 1$

(c) $T = \lambda I$ where I is the identi transformation on $\mathbf{R}^n$

(d) If $\|Tx\| > \|x\|$ for a non zero vect $x \in \mathbf{R}^n$, then T is necessarily singular.

20. The application of Gram-Schmidt process orthonormalization to $u_1 = (1, 1, 0$ $u_2 = (1, 0, 0)$, $u_3 = (1, 1, 1)$ yields

(a) $\frac{1}{\sqrt{2}}(1, 1, 0), (1, 0, 0), (0, 0, 1)$

(b) $\frac{1}{\sqrt{2}}(1, 1, 0), \frac{1}{\sqrt{2}}(1, -1, 0), \frac{1}{\sqrt{2}}(1, 1, 1)$

(c) $(0, 1, 0), (1, 0, 0), (0, 0, 1)$

(d) $\frac{1}{\sqrt{2}}(1, 1, 0), \frac{1}{\sqrt{2}}(1, -1, 0), (0, 0, 1)$

21. Consider the basis $\{u_1, u_2, u_3\}$ of $\mathbf{R}^3$; whe $u_1 = (1, 0, 0)$, $u_2 = (1, 1, 0)$, $u_3 = (1, 1, 1)$. I $\{f_1, f_2, f_3\}$ be the dual basis of $\{u_1, u_2, u$ and f be a linear functional defined $f(a, b, c) = a + b + c$, $(a, b, c) \in \mathbf{R}^3$. $f = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$, then $(\alpha_1, \alpha_2, \alpha_3)$

(a) $(1, 2, 3)$

(b) $(1, 3, 2)$

(c) $(2, 3, 1)$

(d) $(3, 2, 1)$

22. Let $M = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $V = \{M x^t : x \in R^3$

Then an orthonormal basis for V is

(a) $\left\{(1,0,0)^t, \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)^t, \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right.\right.$

(b) $\left\{(1,0,0)^t, \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^t\right\}$

(c) $\left\{(1,0,0)^t, \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^t, \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)^t\right\}$

(d) $\left\{(1,0,0)^t, (0,0,1)^t\right\}$

23. Consider R^3 with the standard inner product. Let $S = \{(1,1,1), (2,-1,2), (1,-2,1)\}$.

For a subset W of R^3, let L(W) denote the linear span of W in R^3. Then an orthonormal set T with $L(S) = L(T)$ is

(a) $\left\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{6}}(1,-2,1)\right\}$

(b) $\{(1,0,0), (0,1,0), (0,0,1)\}$

(c) $\left\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(1,-1,0)\right\}$

(d) $\left\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(0,1,-1)\right\}$

24. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{bmatrix}$ and let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigenvalue of A. The triple $(\lambda_1, \lambda_2, \lambda_3)$ equals

(a) (9, 4, 2)

(b) (8, 4, 3)

(c) (9, 3, 3)

(d) (7, 5, 3)

25. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{bmatrix}$ The matrix P such that

$$P^t A P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$ is

(a) $\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

26. If $H = L_2[0,1]$,

$G = \left\{ f \in I_2 : \int_0^1 x^2 f(x)dx = 0 \right\}$ and $q = x^3$

then orthogonal projection of q onto G is

(a) $x^3 - \frac{6}{5}x^2$

(b) $x^3 - \frac{5}{6}x^2$

(c) $x^3 - \frac{5}{3}x^2$

(d) $x^3 - \frac{5}{2}x^2$

27. Let V be a vector space (over **R**) of dimension 7 and let $f : V \to \mathbf{R}$ be a non-zero linear functional. Let W be a linear subspace of V such that $V = \text{Ker}(f) \oplus W$ where Ker(f) is the null space of f. What is the dimension of W?

(a) 0

(b) 1

(c) 6

(d) None of these

28. Consider the following statements:

1. Let A be a hermitian $N \times N$ positive definite matrix. Then, there exists a hermitian positive definite $N \times N$ matrix B such that $B^2 = A$.
2. Let B be a nonsingular $N \times N$ matrix with real entries. Let B' be its transpose. Then $B'B$ is a symmetric and positive definite matrix.

(a) 1 true 2 false

(b) 2 true 1 false

(c) Boht are true

(d) Both are false

29. Let A be a symmetric $n \times n$ matrix with real entries, which is positive semi-definite, i.e., $x^T Ax \geq 0$ for every (column) vector x, where x^T denotes the (row) vector which is the transpose of x. Pick out the true statements:

(a) The eigenvalues of A are all non-negative

(b) A is invertible;

(c) The principal minor Δ_k of A (i.e. the determinant of the $k \times k$ matrix obtained from the first k rows and first k columns of A) is non-negative for each $1 \leq k \leq n$.

(d) None of these

30. Consider the matrix $M = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Then

(a) M has no real eigen values

(b) All real eigen values of M are positiv

(c) All real eigen values of M are negativ

(d) M has both positive and negative eigen values

31. Let f be a non-zero symmetric bilinear f on $\mathbb{R}^3$. suppose that exist li transformations $T_i : \mathbb{R}^3 \to \mathbb{R}, i = 1,2$ such for all $\alpha,\beta \in \mathbb{R}^3, f(\alpha,\beta) = T_1(\alpha)T_2(\beta)$.

(a) Rank $f = 1$

(b) dim $\{\beta \in \mathbb{R}^3 : f(\alpha,\beta) = 0$ for $\alpha \in \mathbb{R}^3\} = 2$

(c) f is a positive semi-definite or nega semi-definite

(d) $\{\alpha : f(\alpha,\alpha) = 0\}$ is a linear subspac dimension 2.

$\begin{matrix} 2 & 0 \\ 1 & 0 \\ 0 & 2 \\ 2 & 0 \end{matrix}\Bigg)$

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ASSINGMENT-6

1. Let S denote the set of all the prime numbers p with the property that the matrix

$$\begin{bmatrix} 91 & 31 & 0 \\ 29 & 31 & 0 \\ 79 & 23 & 59 \end{bmatrix}$$ has an inverse in the field

$\mathbb{Z}/p\mathbb{Z}$. Then

(a.) $S = \{31\}$

(b.) $S = \{31, 59\}$

(c.) $S = \{7, 13, 59\}$

(d.) S is infinite

2. For a positive integer n, let P_n denote the vector space of polynomials in one variable x with real coefficients and with degree $\leq n$. Consider the map $T : P_2 \to P_4$ defined by $T(p(x)) = p(x^2)$. Then

(a.) T is a linear transformation and dim range$(T) = 5$.

(b.) T is a linear transformation and dim range$(T) = 3$.

(c.) T is a linear transformation and dim range$(T) = 2$.

(d.) T is not a linear transformation.

3. Let A be a real 3×4 matrix of rank 2. Then the rank of $A^t A$, where A^t denotes the transpose A, is:

(a.) exactly 2

(b.) exactly 3

(c.) exactly 4

(d.) at most 2 but not necessarily 2

4. If A is a 5×5 real matrix with trace 15 and if 2 and 3 are eigenvalues of A, each with algebraic multiplicity 2, then the determinant of A is equal to

(a.) 0

(b.) 24

(c.) 120

(d.) 180

5. Let $A \neq I_n$ be an $n \times n$ matrix such that $A^2 = A$, where I_n is the identity matrix of order n. Which of the following statements is false?

(a.) $(I_n - A)^2 = I_n - A$.

(b.) $\text{Trace}(A) = \text{Rank}(A)$.

(c.) $\text{Rank}(A) + \text{Rank}(I_n - A) = n$.

(d.) The eigenvalues of A are each equal to 1.

6. Let A and B be $n \times n$ matrices over $\mathbb{C}$. Then,

(a.) AB and BA always have the same set of eigenvalues.

(b.) If AB and BA have the same set of eigen-values then $AB = BA$.

(c.) If A^{-1} exists then AB and BA are similar.

(d.) The rank of AB is always the same as the rank of BA.

7. Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$ with $b \neq 0$.

(a.) The set of all real solutions of $Ax = b$ is a vector space.

(b.) If u and v are two solutions of $Ax = b$, then $\lambda u + (1 - \lambda)v$ is also a solution of $Ax = b$ for any $\lambda \in \mathbb{R}$.

(c.) For any two solutions u and v of $Ax = b$, the linear combination $\lambda u + (1 - \lambda)v$ is also a solution of $Ax = b$ only when $0 \leq \lambda \leq 1$.

(d.) If rank of A is n, then $Ax = b$ has at most one solution.

8. Let A be an $n \times n$ matrix over $\mathbb{C}$ such that every nonzero vector of $\mathbb{C}^n$ is an eigenvector of A. Then

(a.) All eigenvalues of A are equal.

(b.) All eigenvalues of A are distinct.

(c.) $A = \lambda I$ for some $\lambda \in \mathbb{C}$, where I is the $n \times n$ identity matrix.

(d.) If χ_A and m_A denote the characteristic polynomial and the minimal polynomial respectively, then $\chi_A = m_A$.